RENORMALIZATION, THERMODYNAMIC FORMALISM AND QUASI-CRYSTALS IN SUBSHIFTS.

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Abstract. We examine thermodynamic formalism for a class of renormalizable dynamical systems which in the symbolic space is generated by the Thue-Morse substitution, and in complex dynamics by the Feigenbaum-Coullet-Tresser map. The basic question answered is whether fixed points V of a renormalization operator $\mathcal{R}$ acting on the space of potentials are such that the pressure function $\gamma \mapsto \mathcal{P}(-\gamma V)$ exhibits phase transitions. This extends the work by Baraviera, Leplaideur and Lopes on the Manneville-Pomeau map, where such phase transitions were indeed detected. In this paper, however, the attractor of renormalization is a Cantor set (rather than a single fixed point), which admits various classes of fixed points of $\mathcal{R}$, some of which do and some of which do not exhibit phase transitions. In particular, we show it is possible to reach, as a ground state, a quasi-crystal before temperature zero by freezing a dynamical system.

1. Introduction

1.1. **Background.** Phase transitions are a central theme in statistical mechanics and probability theory. In the physics/probability approach the dynamics is not very relevant and just emerges as a by-product of the invariance by translation. The main difficulty is the geometry of the $\mathbb{Z}^d$ lattice. Considering an interacting particle systems such as the Ising model (see *e.g.* [13, 15]), it is possible to find a measure (called Gibbs measure) that maximizes the probability of obtaining a configuration with minimal free energy associated to a Hamiltonian. This is done considering a finite box and fixing the conditions on its boundary. Then letting the size of the box tend to infinity, the sequence of Gibbs measures have a set of accumulation points. If this set varies non-continuously with respect to the parameters (including the temperature), then the system is said to exhibit a *phase transition.*

In contrast, the time evolution of the system is the central theme in dynamics systems. The theory of thermodynamic formalism has been imported into hyperbolic dynamics in the 70's, essentially by Sinai, Ruelle and Bowen. Gradually, authors started to extend this theory to the non-uniformly hyperbolic case, sometimes applying *inducing techniques* that are also important in this paper. Initially, phase transitions have been less central in dynamical systems, but the development of the theory of ergodic optimization since the 2000's has naturally led mathematicians to introduce (or rather rediscover) the notion of ground states. The question of phase transitions arises naturally in this context.

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Note that vocabulary used in statistical mechanics is sometimes quite different from that used in dynamical system. What in statistical mechanics vocabulary is called a "freezing" transition, such as occur in Fisher-Felderhof models (see *e.g.* [12]), corresponds in the mathematical vocabulary to the Manneville-Pomeau map or the shift with Hofbauer potential (see *e.g.* [28] or [26, Exercise 5.8 on page 98] and also [14]).

Renormalization is an over-arching theme in physics and dynamics, including thermodynamic formalism, see [7] for modern results in the direction. The system that we study in this paper is related to cascade of doubling period phenomenon and the infinitely renormalizable maps *à la* Feigenbaum-Coullet-Tresser, which is on the boundary of chaos (see *e.g.* [27]). Instead of the freezing transitions, the system has its equilibrium state (at phase transition) supported on a Cantor set rather than in a fixed point or a periodic orbit. Stated in physics terminology, we prove that it is possible to reach a quasi-crystal as a ground state before temperature zero by freezing a dynamical system (see Theorems 4 and 5). This issue is related to a question due to van Enter (see [9]). The original question was for $\mathbb{Z}^2$-actions, but we hope that ideas here may be exported to this more complicated case.

Returning to the mathematical motivation, the present paper takes the work of [2] a step further. We investigate the connections between phase transition in the full 2-shift, renormalization for potentials, renormalization for maps (in complex dynamics) and substitutions in the full 2-shift. Here the attractor of renormalization is a Cantor set, rather than a single point, and its thermodynamic properties turn out to be strikingly different.

We recall that Bowen's work [4] on thermodynamic formalism showed that every subshift of finite type with Hölder continuous potential ϕ admits a unique equilibrium state (which is a Gibbs measure). Moreover, the pressure function $\gamma \mapsto \mathcal{P}(-\gamma\phi)$ is real analytic and there are no phase transitions. This is also known as the Griffiths-Ruelle theorem. Hofbauer [17] was the first (in the dynamical systems world) to find continuous non-Hölder potentials for the full two-shift (Σ, σ) allowing a phase transition at some $t = t_0$.

A geometric interpretation of Hofbauer's example leads naturally to the Manneville-Pomeau map $f_{\mathrm{MP}} : [0,1] \to [0,1]$ defined as

$$f_{\mathrm{MP}}(x) = \begin{cases} \frac{x}{1-x} & \text{if } x \in [0, \frac{1}{2}], \\ 2x - 1 & \text{if } x \in (\frac{1}{2}, 1], \end{cases}$$

with a neutral fixed point at 0. This map admits a local renormalization $\psi(x) = \frac{x}{2}$ which satisfies

$$f_{\mathrm{MP}}^2 \circ \psi(x) = \psi \circ f_{\mathrm{MP}}(x) \qquad \text{for all } x \in [0, \frac{1}{2}]. \tag{1}$$

If we differentiate Equation (1), take logarithms and subtract $\log \psi' \equiv \log \frac{1}{2}$ from both sides of the equality, we find

$$\log |f'_{\mathrm{MP}}| = \log |f'_{\mathrm{MP}}| \circ f_{\mathrm{MP}} \circ \psi(x) + \log |f'_{\mathrm{MP}}| \circ \psi(x). \tag{2}$$

Passing to the shift-space again (via the itinerary map for the standard partition $\{[0,\frac{1}{2}],\ (\frac{1}{2},1]\}$), we are naturally led to renormalization in the shift. Of prime importance are the solutions of the equation

$$\sigma^2 \circ H = H \circ \sigma, \tag{3}$$

which replaces the renormalization scaling ψ in (1). Equation (2) leads to an operator $\mathcal{R}$ defined by

$$\mathcal{R}(V) = V \circ \sigma \circ H + V \circ H.$$

In [2], the authors investigated the case of the substitution

$$H_{\mathrm{MP}} : \begin{cases} 0 \to 00, \\ 1 \to 01, \end{cases}$$

which has a unique fixed point 0^∞, corresponding to the neutral fixed point 0 of f_{MP}. In [2], the map H_{MP} was not presented as a substitution but we emphasize here (and it is an improvement because it allows more general studies) that it indeed is; more generally, any constant-length k substitution solves Equation (3) (with σ^k instead of σ^2). It is also shown in [2] that the operator $\mathcal{R}$ fixes the Hofbauer potential

$$V(x) := \log\frac{n+1}{n} \qquad \text{if } x \in [0^n1] \setminus [0^{n+1}1], \quad n > 0.$$

Moreover, the lift of $\log f'_{\mathrm{MP}}$ belongs to the *stable set* of the Hofbauer potential. This fact is somewhat mysterious because the substitution H_{MP} *is not* the lift of the scaling function $\psi : x \mapsto x/2$.

In this paper we focus on the Thue-Morse substitution; see (4) for the definition. It is one of the simplest substitutions satisfying the renormalization equality (3) and contrary to H_{MP}, the attractor for the Thue-Morse substitution, say $\mathbb{K}$, is not a periodic orbit but a Cantor set. Yet similarly to the Manneville-Pomeau fixed point, $\sigma : \mathbb{K} \to \mathbb{K}$ has zero entropy and is uniquely ergodic. This is one way to define quasi-crystal in ergodic theory.

The thermodynamic formalism for the Thue-Morse substitution is much more complicated, and more interesting, than for the Manneville-Pomeau substitution. This is because Cantor structure of the attractor admits a more intricate recursion behavior of nearby points (although it has zero entropy) characterized by what we call "accidents" in Section 2.3, which are responsible for the lack of phase transitions for the "good" fixed point for $\mathcal{R}$, This allows much more chaotic shadowing than when the attractor of the substitution is a periodic orbit. We want to emphasize here that our results are extendible to more general substitutions, but to get the main ideas across, we focus on the Thue-Morse shift in this paper.

1.2. **Statements of results.**

The Thue-Morse substitution

$$H := H_{\mathrm{TM}} : \begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases} \tag{4}$$

has two fixed points

$$\rho_1 = 1001\ 0110\ 1001\ 0110\ 01\ldots \qquad \text{and} \qquad \rho_0 = 0110\ 1001\ 0110\ 1001\ 10\ldots$$

Let $\mathbb{K} = \overline{\cup_n \sigma^n(\rho_0)} = \overline{\cup_n \sigma^n(\rho_1)}$ be the corresponding subshift of the full shift (Σ, σ) on two symbols. The renormalization equation (3) holds in Σ: $H \circ \sigma = \sigma^2 \circ H$, and we define the *renormalization operator* acting on functions $V : \Sigma \to \mathbb{R}$ as

$$(\mathcal{R}V)(x) = V \circ \sigma \circ H(x) + V \circ H(x).$$

We consider the usual metric on Σ: $d(x,y) = \frac{1}{2^n}$ if $n = \min\{i \geqslant 1 : x_i \neq y_i\}$. This distance is sometimes graphically represented as follows:

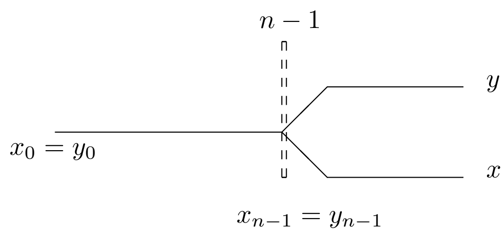


FIGURE 1. The sequence x and y coincide for digits 0 up to $n-1$ and then split.

Note that $d(H^n x, H^n y) = d(x,y)^{2^n}$: if x and y coincide for m digits, then $H^n(x)$ and $H^n(y)$ coincide for $2^n m$ digits.

The first two results deal with the continuous fixed points for the renormalization operator $\mathcal{R}$. The main issue is to determine fixed points and their *weak stable leaf*, namely the potentials attracted by the considered fixed point by iterations of $\mathcal{R}$.

The second series of results deals with the thermodynamical formalism; we study if some class of potentials related to weak stable leaf of the fixed points, exhibit a phase transition. In particular, Theorem 5 is related to a question of Van Enter et al. (see *e.g.* [9, 10]) asking whether it is possible to reach a quasi-crystal by freezing a system before zero temperature.

The last result (Theorem 6) returns to the geometrical dynamics and shows the main difference between the Thue-More case and the Manneville-Pomeau case. Due to the Cantor structure of the attractor of the substitution, there exist non-continuous but locally constant (on $\mathbb{K}$ up to a finite number of points) fixed points for $\mathcal{R}$. As the Hofbauer potential represents the logarithm of the derivative of an affine approximation of the Manneville-Pomeau map, one of these potentials, V_u, represents the logarithm of the derivative of an affine approximation to the Feigenbaum-Coullet-Tresser map $f_{feig} : \mathbb{C} \to \mathbb{C}$. The main difference with the Manneville-Pomeau case is that here, V_u has no phase transition whereas $-\log|f'_{feig}|$ has.

1.2.1. *Results on continuous fixed points for* $\mathcal{R}$. Define the one-parameter family of potentials

$$U_c = \begin{cases} c & \text{on } [01], \\ -c & \text{on } [10], \\ 0 & \text{on } [00] \cup [11]. \end{cases} \tag{5}$$

It is easy to verify that U_c is a fixed point of $\mathcal{R}$. Given a fixed function $V : \Sigma \to \mathbb{R}$, the *variation on k-cylinders* $\mathrm{Var}_k(V)$ is defined as

$$\mathrm{Var}_k(V) := \max\{|V(x) - V(y)|,\ x_j = y_j \text{ for } j = 0, \dots, k-1\}.$$

The condition $\sum_{k=1}^{\infty} \mathrm{Var}_k(W) < \infty$ holds if *e.g.* W is Hölder continuous.

Theorem 1. *If W is a continuous fixed point of $\mathcal{R}$ on $\mathbb{K}$ such that*

$$\sum_{k=1}^{\infty} Var_k(W) < \infty,$$

then $W = U_c$ for $c = W(\rho_0)$.

As for the Hofbauer case, we produce a non-negative continuous fixed point for $\mathcal{R}$ with a well-defined *weak stable set*[1]

Theorem 2. *There exists a unique function $\widetilde{V}$, such that $\widetilde{V} = \lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{n-1} \mathcal{R}^k V$ for every continuous V satisfying $V(x) = \frac{1}{n} + o(\frac{1}{n})$ if $d(x,\mathbb{K}) = 2^{-n}$. Moreover $\widetilde{V}$ is $\mathcal{R}$-invariant, continuous and positive except on $\mathbb{K}$: $\frac{1}{2n} \leqslant \widetilde{V}(x) \leqslant \frac{1}{n-1}$ if $d(x,\mathbb{K}) = 2^{-n}$.*

1.2.2. *Results on Thermodynamic Formalism.* We refer to Bowen's book [5] for the background on thermodynamic formalism, equilibrium states and Gibbs measures in Σ. However, in contrast to Bowen's book, our potentials are not Hölder-continuous.

For a given potential $W : \Sigma \to \mathbb{R}$, the pressure of W is defined by

$$\mathcal{P}(W) := \sup\{h_\mu(\sigma) + \int W\, d\mu\},$$

where $h_\mu(\sigma)$ is the Kolmogorov entropy of the invariant probability measure μ. The supremum is a maximum in Σ whenever W is continuous. Any measure realizing this maximum is called an equilibrium state. We want to study the regularity of the function $\gamma \mapsto \mathcal{P}(-\gamma W)$. For simplicity, this function will also be denoted by $\mathcal{P}(\gamma)$. If $\mathcal{P}(\gamma)$ fails to be analytic, we speak of a phase transition. We are in particular interested in the special phase transition as $\gamma \to \infty$: easy and classical computations show that $\mathcal{P}(\gamma)$ has an asymptote of the form $-a\gamma + b$ as $\gamma \to \infty$. By an *ultimate phase transition* we mean that $\mathcal{P}(\gamma)$ reaches its asymptote at some γ'. In this case, there cannot be another phase transition for larger γ, hence *ultimate*. Then, by a convexity argument, $\mathcal{P}(\gamma) = -a\gamma + b$ for any $\gamma \geqslant \gamma'$. One of the main motivations for studying ultimate phase transitions is that the quantity a satisfies

$$a = \inf\left\{\int W\, d\mu,\ \mu \text{ is a shift-invariant probability measure}\right\}.$$

An example of an ultimate phase transition for rational maps can be found in [22]. The Manneville-Pomeau map is another classical example.

Theorem 3 (No phase transition)**.** *Let $a > 1$ and $V : \Sigma \to \mathbb{R}$ be a continuous function satisfying $V(x) = \frac{1}{n^a} + o(\frac{1}{n^a})$ if $d(x,\mathbb{K}) = 2^{-n}$. Then, for every $\gamma \geqslant 0$, there exists a unique equilibrium state associated to $-\gamma V$ and it gives positive mass to every open set. The pressure function $\gamma \mapsto \mathcal{P}(\gamma)$ is analytic and positive on $[0,\infty)$, although it converges to zero as $\gamma \to \infty$.*

Theorem 4 (Phase transition)**.** *Let $a \in (0,1)$ and $V : \Sigma \to \mathbb{R}$ be a continuous function satisfying $V(x) = \frac{1}{n^a} + o(\frac{1}{n^a})$ if $d(x,\mathbb{K}) = 2^{-n}$. Then there exists γ_1 such that for every $\gamma > \gamma_1$ the unique equilibrium state for $-\gamma V$ is the unique invariant measure $\mu_{\mathbb{K}}$ supported on $\mathbb{K}$. For $\gamma < \gamma_1$, there exists a unique equilibrium state associated to $-\gamma V$ and it gives*

[1] In [2] it was proven that $\mathcal{R}^n(V)$ converges to the fixed point $\widetilde{V}$; here we only get convergence in the Cesaro sense.

positive mass to every open set in Σ. The pressure function $\gamma \mapsto \mathcal{P}(\gamma)$ is positive and analytic on $[0, \gamma_1)$.

These results show that case $a = 1$ (*i.e.,* the Hofbauer potential) is the border between the regimes with and without phase transition. Whether there is a phase transition for the case $a = 1$ (*i.e.,* the fixed point $\tilde{V}$) or in other words the analog of the Hofbauer potential, discussed in [2], is much more subtle. We intend to come back to this question in a later paper.

The full shift (Σ, σ) can be interpreted geometrically by a degree 2 covering of the circle. The Manneville-Pomeau map can be viewed this way; it is expanding except for a single (one-sided) indifferent fixed point. When dealing with the Thue-Morse shift, it is natural to look for a circle covering with an indifferent Cantor set.

Theorem 5. *There exist $\mathcal{C}^1$ maps $f_a : [0,1] \circlearrowleft$, semi-conjugate to the full 2-shift and expanding everywhere except on a Cantor set $\widetilde{\mathbb{K}}$, such that $\widetilde{\mathbb{K}}$ is conjugate to $\mathbb{K}$ in Σ and if $a \in (0, 1)$, then $-\gamma \log f_a'$ has an ultimate phase transition.*

Another geometric realization of the Thue-Morse shift and the prototype of renormalizability in one-dimensional dynamics is the Feigenbaum map. This quadratic interval map $f_{\text{q-feig}}$ has zero entropy, but when complexified it has entropy $\log 2$. Moreover, it is conjugate to another analytic degree 2 covering map on $\mathbb{C}$, which we call f_{feig}, that is fixed by the Feigenbaum renormalization operator

$$\mathcal{R}_{\text{feig}} f = \Psi^{-1} \circ f^2 \circ \Psi$$

where Ψ is linear f-dependent holomorphic contraction. Arguments from complex dynamics give that $\mathcal{P}(-\gamma \log |f_{\text{feig}}'|) = 0$ for all $\gamma \geqslant 2$, see Proposition 23. Because $h_{top}(f_{\text{feig}}) = \log 2$ on its Julia set, the potential $-\gamma_1 \log |f_{\text{feig}}'|$ has a phase transition for some $\gamma_1 \in (0, 2]$. When lifted to symbolic space, $-\log |f_{\text{feig}}'|$ produces an unbounded potential V_{feig} which is fixed by $\mathcal{R}$. We can find a potential V_u, which is constant on

$$(\sigma \circ H)^k(\Sigma) \setminus (\sigma \circ H)^{k+1}(\Sigma)$$

for each k such that $\|V_{\text{feig}} - V_u\|_\infty < \infty$ and analyze the thermodynamic properties of V_u. Although $\mathcal{P}(-\gamma_1 V_{\text{feig}}) = 0$ for some $\gamma_1 \leqslant 2$, it is surprising to see that the potential V_u exhibits no phase transition. We emphasize here an important difference with the Manneville-Pomeau case, where both the potential $-\gamma \log |f_{MP}'|$ and its countably piecewise version, the Hofbauer potential, which is constant on cylinder sets $(H_{MP})^k(\Sigma) \setminus (H_{MP})^{k+1}(\Sigma) = [0^{2k+1}1]$, undergo a phase transition.

Theorem 6 (No phase transition for unbounded fixed point V_u)**.** *The unbounded potential V_u given by*

$$V_u(x) = \alpha(k-1) \quad \textit{for} \quad x \in (\sigma \circ H)^k(\Sigma) \setminus (\sigma \circ H)^{k+1}(\Sigma)$$

is a fixed point of $\mathcal{R}$ for any $\alpha \in \mathbb{R}$. If $\alpha < 0$, then for every $\gamma \geqslant 0$, there exists a unique equilibrium state for $-\gamma V_u$. It gives positive mass to any open set in Σ. The pressure function $\gamma \mapsto \mathcal{P}(-\gamma V_u)$ is analytic and positive for all $\gamma \in [0, \infty)$.

The exact definition of the equilibrium state for this unbounded potential can be found in Subsection 3.5.

1.3. **Outline of the paper.** In Section 2 we prove Theorems 1 and 2. In the first subsection we recall some and prove other results on the Thue-Morse substitution and its associated attractor $\mathbb{K}$.

In Section 3 we study the thermodynamic formalism and prove Theorems 4, 5 and 6. This section uses extensively the theory of local thermodynamic formalism defined in [19] and developed in further works of the author. Finally, in the Appendix, we explain the relation between the Thue-Morse shift and the Feigenbaum map, and state and prove Proposition 23.

2. Renormalization in the Thue-Morse shift-space

2.1. **General results on the Thue-Morse shift-space.** Let $\sigma : \Sigma \to \Sigma$ be the full shift on $\Sigma = \{0,1\}^{\mathbb{N}}$. If $x = x_0x_1x_2x_4 \cdots \in \Sigma$, let $[x_0 \dots x_{n-1}]$ denote the n-cylinder containing x, and let $\bar{x}_i = 1 - x_i$ be our notation for the opposite symbol.

Recall that ρ_0 and ρ_1 are the fixed points of the Thue–More substitution, and that $\mathbb{K} = \overline{\mathrm{orb}_\sigma(\rho_0)} = \overline{\mathrm{orb}_\sigma(\rho_1)}$ is a uniquely ergodic and zero-entropy subshift. We denote by $\mu_{\mathbb{K}}$ its invariant measure.

We give here some properties for the Thue-Morse sequence that can be found in [3, 6, 8, 21].

(1) *Left-special* words (*i.e.,* words w such that both $0w$ and $1w$ appear in $\mathbb{K}$) are prefixes of $H^k(010)$ or of $H^k(101)$ for some $k \geqslant 0$.
(2) *Right-special* words (*i.e.,* words w such that both $w0$ and $w0$ appear in $\mathbb{K}$) are suffixes of $H^k(010)$ or of $H^k(101)$ for some $k \geqslant 0$.
(3) *Bispecial* words (*i.e.,* words w such that both left and right-special) are precisely the words $\tau_k := H^k(0)$, $\bar{\tau}_k := H^k(1)$, $\tau_k\bar{\tau}_k\tau_k = H^k(010)$ and $\bar{\tau}_k\tau_k\bar{\tau}_k = H^k(101)$ for $k \geqslant 0$. There are four ways in which a word w can be extended to awb, *i.e.,* with a symbol both to the right and left. It is worth noting that for $w = \tau_k$ or $\bar{\tau}_k$, all four ways indeed occur in $\mathbb{K}$, while for $w = \tau_k\bar{\tau}_k\tau_k$ or $\bar{\tau}_k\tau_k\bar{\tau}_k$ only two extensions occur.
(4) The Thue-Morse sequence has low *word-complexity*:
$$p(n) = \begin{cases} 3 \cdot 2^m + 4r & \text{if } 0 \leqslant r < 2^{m-1}, \\ 4 \cdot 2^m + 2r & \text{if } 2^{m-1} \leqslant r < 2^m, \end{cases}$$
where $n = 2m + r + 1$.
(5) The Thue-Morse shift is almost square-free in the sense that if $w = w_1 \dots w_n$ is some word, then ww can appear in $\mathbb{K}$, but not www_1. The nature of the Thue-Morse substitution is such that ρ_0 and ρ_1 are concatenations of the words τ_k and $\bar{\tau}_k$. Appearances of τ_k and $\bar{\tau}_k$ can overlap, but not for too long compared to their lengths, as made clear in Corollary 4

The next lemma shows that almost-invertibility of σ on $\mathbb{K}$ implies some shadowing close to $\mathbb{K}$.

Lemma 1. *For $x \in \Sigma$ with $d(x, \mathbb{K}) < 2^{-5}$, let $y, y' \in \mathbb{K}$ be the closest points in $\mathbb{K}$ to x and $\sigma(x)$ respectively. If $y' \neq \sigma(y)$, then y' starts as τ_k, $\bar{\tau}_k$, $\tau_k\bar{\tau}_k\tau_k$ or $\bar{\tau}_k\tau_k\bar{\tau}_k$ for some $k \geqslant 3$.*

Proof. As $y' \neq \sigma(y)$, there is another $z \in \mathbb{K}$ such that $\sigma(z) = y'$ and $z_0 \neq y_0 = x_0$. Let d be maximal such that $y_1 \dots y_{d-1} = z_1 \dots z_{d-1}$, so $y_d \neq z_d$. This means that the word $y_1 \dots y_{d-1}$ is bi-special, and according to property (3) has to coincide with τ_k, $\bar{\tau}_k$, $\tau_k\bar{\tau}_k\tau_k$ or $\bar{\tau}_k\tau_k\bar{\tau}_k$ for some $k \geqslant 3$. □

Due to the Cantor structure of $\mathbb{K}$, the distance of an orbit to $\mathbb{K}$ is not a monotone function in the time. This is the main problem we will have to deal with.

Definition 2. *Let $x \in \Sigma$ be such that $d(\sigma(x), \mathbb{K}) < 2d(x, \mathbb{K})$. Then we say that we have an* accident *at $\sigma(x)$. By extension, if $d(\sigma^{k+1}(x), \mathbb{K}) = 2d(\sigma^k(x), \mathbb{K})$ for every $k < n - 1$, but $d(\sigma^n(x), \mathbb{K}) < 2d(\sigma^{n-1}(x), \mathbb{K})$, then we say that we have an* accident *at time n.*

Proposition 3. *Assume that $-\log_2 d(x, \mathbb{K}) = d$ and that $b \leqslant d$ is the first accident for the piece of orbit $x, \dots, \sigma^d(x)$, then*

- *$x_b x_{b+1} \dots x_{d-1}$ is a bispecial word for $\mathbb{K}$;*
- *$d - b = 3^\varepsilon \cdot 2^k$ for some k and $\varepsilon \in \{0, 1\}$;*
- *$x_0 \dots x_{d-1}$ is neither right-special nor left-special;*
- $b \geqslant \begin{cases} 2^k & \text{if } d - b = 2^k, \\ 2^{k+1} & \text{if } d - b = 3 \cdot 2^k. \end{cases}$

Proof. Let y and $y' \in \mathbb{K}$ be such that x and y coincide for d digits and $\sigma^b(x)$ and y' coincide for at least $d - b$ digits. Then

$$x_b x_{b+1} \dots x_{d-1} = y_b y_{b+1} \dots y_{d-1} = y'_0 y'_1 \dots y'_{d-b-1}$$

is a right-special because it can be continued both as y' and y. The word $x_b x_{b+1} \dots x_{d-1}$ is also a left-special, because otherwise, by Lemma 1, only one preimage of y' by σ would be in $\mathbb{K}$ and this would coincide with the word $y_{b-1} y_b \dots y_{d-b-1}$. Then $b-1$ rather than b would be the first accident. By property (3) above, $d - b = 3^\varepsilon 2^k$. On the other hand, $x_0 \dots x_{d-1}$ cannot be right-special, because otherwise there would be a point $\tilde{x} = x_0 \dots x_{d-1}\bar{x}_d \dots \in \mathbb{K}$ with $d(x, \tilde{x}) < 2^{-d}$. If $x_0 \dots x_{d-1}$ is left-special, then

To finish the proof of the proposition we need to check that the next accident cannot happen too early. Assume that $x_0 \dots x_{d-2}$ start as $\rho_0 = r_0 r_1 r_3 \dots$ (the argument for ρ_1 is the same). Let $\pi(n) = \#\{0 \leqslant i < n : r_i = 1\} - \#\{0 \leqslant i < n : r_i = 0\}$ count the surplus of 1's within the first n entries of ρ_1. Clearly $\pi(n) = 0$ for even n and $\pi(n) = \pm 1$ otherwise. Assume the word τ_k starts in ρ_0 at some digit $m < 2^k$. If $\pi(m) = 1$, then $\pi(m + 3) = 2$ while if $\pi(m) = -1$, then $\pi(m + 7) = -2$. A similar argument works if $\bar{\tau}_k$ stars at digit m. This shows that if τ_k or $\bar{\tau}_k$ can only start in ρ_0 at even digits. This means that we can take the inverse H^{-1} and find that τ_{k-1} (or $\bar{\tau}_k$) start at digit $m/2 < 2^{k-2}$ in ρ_0. Repeating this argument, we arrive at τ_3 or $\bar{\tau}_3$ starting before digit $8 = 2^{4-1}$ of ρ_0, which is definitely false, as we can see by inspecting $\rho_0 = 0110\ 1001\ 1001\ 0110\ \dots$. Note also that the bound 2^k is sharp, because $\bar{\tau}_k$ starts in ρ_0 at entry 2^k.

Finally, we need to answer the same question for bispecial words $\tau_k\bar{\tau}_k\tau_k = \tau_{k+1}\tau_k$ and $\bar{\tau}_k\tau_k\bar{\tau}_k = \bar{\tau}_{k+1}\bar{\tau}_k$. The previous argument shows that neither can start before digit 2^{k+1}, and also this bound is sharp, because $\bar{\tau}_k\tau_k\bar{\tau}_k$ starts in ρ_0 at entry 2^{k+1}. □

Corollary 4. *Occurrences of τ_k and $\bar{\tau}_k$ cannot overlap for more than 2^{k-1} digits.*

Proof. We consider the prefix τ_k of ρ_0 only, as the other case is symmetric. If the overlap was more than 2^{k-1} digits, then τ_{k-1} or $\bar{\tau}_{k-1}$ would appear in ρ_0 before digit 2^{k-1}, which contradicts part (3) of Proposition 3 □

Lemma 5. *For each $k \geqslant 1$, the Thue-Morse substitution H satisfies $\mathbb{K} = \bigsqcup_{j=0}^{2^k-1} \sigma^j \circ H^k(\mathbb{K})$, where $\sqcup$ indicates disjoint union, so $\sigma^i \circ H^k(\mathbb{K}) \cap \sigma^j \circ H^k(\mathbb{K}) = \emptyset$ for all $0 \leqslant i < j < 2^k$.*

Proof. Take $x \in \mathbb{K}$, so there is a sequence $(n_k)_{k\in\mathbb{N}}$ such that $x = \lim_k \sigma^{n_k}(\rho_0)$. If this sequence contains infinitely many even integers, then $x = \lim_k \sigma^{2m_k}(\rho_0) = \lim_k \sigma^{2m_k} \circ H(\rho_0) = \lim_k H \circ \sigma^{m_k}(\rho_0) \in H(\mathbb{K})$. Otherwise, $(n_k)_{k\in\mathbb{N}}$ contains infinitely many odd integers and $x = \lim_k \sigma^{1+2m_k}(\rho_0) = \lim_k \sigma \circ \sigma^{2m_k} \circ H(\rho_0) = \lim_k \sigma \circ H \circ \sigma^{m_k}(\rho_0) \in \sigma \circ H(\mathbb{K})$. Therefore $\mathbb{K} \subset H(\mathbb{K}) \cup \sigma \circ H(\mathbb{K})$.

Now if $x = H(a) = \sigma \circ H(b) \in \mathbb{K}$, then

$$x = a_0\bar{a}_0a_1\bar{a}_1a_2\bar{a}_2\dots = \bar{b}_0b_1\bar{b}_1b_2\bar{b}_2\dots\ ,$$

so $\bar{b}_0 = a_0 \neq \bar{a}_0 = b_1 \neq \bar{b}_1 = a_1 \neq \bar{a}_1 = b_2 \neq b_2 = a_2$. Therefore $x = 101010\dots$ or $010101\dots$, but neither belongs to $\mathbb{K}$.

Now for the induction step, assume $\mathbb{K} = \bigsqcup_{j=0}^{2^k-1} \sigma^j \circ H^k(\mathbb{K})$. Then since H is one-to-one,

$$\begin{aligned}
\mathbb{K} &= \bigsqcup_{j=0}^{2^k-1} \sigma^j \circ H^k(H(\mathbb{K}) \sqcup \sigma \circ H(\mathbb{K})) \\
&= \left(\bigsqcup_{j=0}^{2^k-1} \sigma^j \circ H^{k+1}(\mathbb{K})\right) \bigsqcup \left(\bigsqcup_{j=0}^{2^{k-1}} \sigma^j \circ H^k \circ \sigma \circ H(\mathbb{K}))\right) \\
&= \left(\bigsqcup_{j=0}^{2^k-1} \sigma^j \circ H^{k+1}(H(\mathbb{K})\right) \bigsqcup \left(\bigsqcup_{j=0}^{2^k-1} \sigma^{j+2^k} \circ H^{k+1}(\mathbb{K}))\right) \\
&= \bigsqcup_{j=0}^{2^{k+1}-1} \sigma^j \circ H^{k+1}(\mathbb{K}).
\end{aligned}$$

□

Lemma 6. *Let x be in the cylinder $[ab]$ with $a, b \in \{0,1\}$. Then the accumulation point of $(\sigma \circ H)^k(x)$ are $0\rho_b$ and $1\rho_b$. More precisely, the $(\sigma \circ H)^{2k}(x)$ converges to $a\rho_b$ and $(\sigma \circ H)^{2k+1}(x)$ converges to $\bar{a}\rho_b$.*

Proof. By definition of H we get $H(x) = a\bar{a}H(b)\dots$ Hence $\sigma \circ H(x) = \bar{a}H(b)\dots$ By induction we get

$$(\sigma \circ H)^{2k}(x) = aH^{2k}(b)\dots \quad \text{and} \quad (\sigma \circ H)^{2k+1}(x) = \bar{a}H^{2k+1}(b).$$

Therefore $H^n(b)$ converges to ρ_b, for $b = 0, 1$. □

2.2. **Continuous fixed points of $\mathcal{R}$ on $\mathbb{K}$: Proof of Theorem 1.** We recall that we have $\mathcal{R}(V) = V \circ \sigma \circ H + V \circ H$. Therefore

$$\begin{aligned}\mathcal{R}^2 V &= \mathcal{R}(V \circ \sigma \circ H + V \circ H)\\ &= V \circ \sigma \circ H \circ \sigma \circ H + V \circ \sigma \circ H^2 + V \circ H \circ \sigma \circ H + V \circ H^2\\ &= V \circ \sigma^3 \circ H^2 + V \circ \sigma^2 \circ H^2 + V \circ \sigma \circ H^2 + V \circ H^2,\end{aligned}$$

and in general

$$\mathcal{R}^n V = S_{2^n} V \circ H^n \quad \text{where} \quad (S_k V)(x) = \sum_{i=0}^{k-1} V \circ \sigma^i(x)$$

is the k-th ergodic sum.

Lemma 7. *If $V \in L^1(\mu_{\mathbb{K}})$ is a fixed point of $\mathcal{R}$, then $\int_{\mathbb{K}} V \, d\mu_{\mathbb{K}} = 0$.*

Proof. For any typical (w.r.t. Birkhoff's Ergodic Theorem) $y \in \mathbb{K}$ we get

$$V(y) = (\mathcal{R}^n V)(y) = \sum_{j=0}^{2^n-1} V \circ \sigma^j \circ H^n(y).$$

Hence

$$\frac{1}{2^n} V(y) = \frac{1}{2^n} \sum_{j=0}^{2^n-1} V \circ \sigma^j \circ H^n(y).$$

The left hand side tend to 0 as $n \to \infty$ and the right hand side tends to $\int_{\mathbb{K}} V \, d\mu_{\mathbb{K}}$. $\square$

Lemma 8. *Let W be any continuous fixed point for $\mathcal{R}$ (on $\mathbb{K}$). Then, for $j = 0, 1$,*

$$W(01\rho_j) + W(10\rho_j) = 0 \quad \text{and} \quad W(1\rho_j) = W(10\rho_j) + W(0\rho_j).$$

Proof. Using the equality $W(x) = (\mathcal{R}W)(x) = W \circ H(x) + W \circ \sigma \circ H(x)$ we immediately get:

$$W \circ (\sigma \circ H)^n(x) = W \circ H \circ (\sigma \circ H)^n(x) + W \circ (\sigma \circ H)^{n+1}(x).$$

Using Lemma 6 on this new equality, we obtain

$$W(i\rho_j) = W(i\bar{i}\rho_j) + W(\bar{i}\rho_j),$$

for $i, j \in \{0, 1\}$. This gives the second equality of the lemma (for $i = 1$). The symmetric formula is obtained from the case $i = 0$, and then adding both formulas yields $W(01\rho_j) + W(10\rho_j) = 0$. $\square$

Remark 1. *Lemma 8 still holds if the potential is only continuous at points of the form $i\rho_j$ and $i\bar{i}\rho_j$ with $i, j \in \{0, 1\}$.* $\blacksquare$

Recall the one-parameter family of potentials U_c from (5). They are fixed points of $\mathcal{R}$, not just on $\mathbb{K}$, but globally on Σ. Let $i : \Sigma \to \Sigma$ be the involution changing digits 0 to 1 and vice versa. Clearly $U_c = -U_c \circ i$. We can now prove Theorem 1.

Proof of Theorem 1. Let W be a potential on $\mathbb{K}$, that is fixed by $\mathcal{R}$. We assume that the variations are summable: $\sum_{k=1}^{\infty} \mathrm{Var}_k(W) < \infty$.

We show that W is constant on 2-cylinders. Let $x = x_0x_1\dots$ and $y = y_0y_1\dots$ be in the same 2-cylinder (namely $x_0 = y_0$ and $x_1 = y_1$). Then, for every n, $H^n(x)$ and $H^n(y)$ coincide for (at least) 2^{n+1} digits. Therefore

$$\begin{aligned}|W(x) - W(y)| &= |(\mathcal{R}^nW)(x) - (\mathcal{R}^nW)(y)| \\ &= |(S_{2^n}W)(H^n(x)) - (S_{2^n}W)(H^n(y))| \\ &\leqslant \sum_{k=2^n+1}^{2^{n+1}} \mathrm{Var}_k(W).\end{aligned}$$

Convergence of the series $\sum_k \mathrm{Var}_k(W)$ implies that $\sum_{k=2^n+1}^{2^{n+1}} \mathrm{Var}_k(W) \to 0$ as $n \to \infty$. This yields that W is constant on 2-cylinders.

Lemma 8 shows that $W|_{[01]} = -W|_{[10]}$. Again, the second equality in that lemma used for both ρ_0 and ρ_1 shows that $W|_{[00]} = W|_{[11]} = 0$. Therefore $W = U_c$ with $c = W(\rho_0)$, and the proof is finished. □

2.3. **Global fixed points for $\mathcal{R}$: Proof of Theorem 2.** To give an idea why Theorem 2 holds, observe that the property $V(x) = \frac{1}{n} + o(\frac{1}{n})$ if $d(x, \mathbb{K}) = 2^{-n}$ (so V vanishes on $\mathbb{K}$ but is positive elsewhere) is in spirit preserved under iterations of $\mathcal{R}$, provided the shift σ doubles the distance from $\mathbb{K}$. Let $\mathcal{D}$ denote the class of potentials satisfying this property. Choose x such that $d(x, \mathbb{K}) = 2^{-m}$. Taking the limit of Riemann sums, and since $\mathcal{R}$ preserves the class of non-negative functions, we obtain

$$\begin{aligned}0 \leqslant (\mathcal{R}^nV)(x) &= \sum_{j=0}^{2^n-1} \frac{1}{2^nm - j} + \sum_{j=0}^{2^n-1} o(\frac{1}{2^nm - j}) \\ &\to_{n\to\infty} (1 + o(1)) \int_0^1 \frac{1}{m - t}\, dt \\ &= (1 + o(1)) \log \frac{m}{m - 1} = \frac{1}{m} + o(\frac{1}{m}).\end{aligned}$$

However, it may happen that $d(\sigma(y), \mathbb{K}) < 2d(y, \mathbb{K})$ for some $y = \sigma^j \circ H^n(x)$, in which case we speak of an accident (see Definition 2). The proof of the proposition includes an argument that accidents happen only infrequently, and far apart from each other.

Remark 2. *We emphasize an important bi-product of the previous computation. If V is of the form $V(x) = o(\frac{1}{m})$ when $d(x, \mathbb{K}) = 2^{-m}$, then $\mathcal{R}^n(V)$ converges to 0. See also Proposition 9.* ■

Proof of Theorem 2. The proof has three steps. In the first step we prove that the class $\mathcal{D}$ is invariant under $\mathcal{R}$. In the second step we show that $\mathcal{R}^n(V_0)$, with V_0 defined by $V_0(x) = \frac{1}{m}$ if $d(x, \mathbb{K}) = 2^{-m}$, is positive (outside $\mathbb{K}$) and bounded from above. In the last step we deduce from the two first steps that there exists a unique fixed point and that it is continuous and positive. We also briefly explain why it gives the result for any $V \in \mathcal{D}$.

Step 1. We recall that $\mathcal{R}$ is defined by $(\mathcal{R}V)(x) := V \circ H(x) + V \circ \sigma \circ H(x)$. As H and σ are continuous, $\mathcal{R}(V)$ is continuous if V is continuous. Let $x \in \Sigma$, then if $x_K \in \mathbb{K}$ is such that

$$d(x, \mathbb{K}) = d(x, x_K) = 2^{-m}, \text{ then } d(H(x), H(x_K)) = 2^{-2m}. \tag{6}$$

We claim that if $m \geqslant 3$, then $d(H(x), \mathbb{K}) = d(H(x), H(x_K))$. Let us assume by contradiction that $y \in \mathbb{K}$ is such that $d(H(x), \mathbb{K}) = d(H(x), y) < d(H(x), H(x_K))$. By Lemma 5, y belongs either to $H(\mathbb{K})$ or to $\sigma \circ H(\mathbb{K})$. In the first case, say $H(z) = y$, we get

$$d(H(x), H(z)) < d(H(x), H(x_K)).$$

This would yield $d(x, z) < d(x, x_K)$ which contradicts the fact that $d(x, \mathbb{K}) = d(x, x_K)$.

In the other case, say $y = \sigma \circ H(z)$, $m \geqslant 3$ yields $H(x) = a_0\bar{a}_0a_1\bar{a}_1a_2\bar{a}_2\dots$ and $\sigma \circ H(z) = \bar{b}_0b_1\bar{b}_1b_2\bar{b}_2\dots$. As in the proof of Lemma 5 this would show that y must start with 010101 or 101010. However, both are forbidden in $\mathbb{K}$ and this produces a contradiction. This finishes the proof of the claim.

Lemma 1 also shows that $d(\sigma \circ H(x), \mathbb{K}) = 2^{-(2m-1)} = d(\sigma \circ H(x), \sigma \circ H(x_K))$. Therefore

$$(\mathcal{R}V)(x) = V \circ H(x) + V \circ \sigma \circ H(x) = \frac{1}{2m} + \frac{1}{2m-1} + o(\frac{1}{m}) = \frac{1}{m} + o(\frac{1}{m}). \tag{7}$$

Step 2. We establish upper and lower bounds for $\mathcal{R}^n(V_0)$ where V_0 is defined by $V_0(x) = \frac{1}{m}$ if $d(x, \mathbb{K}) = 2^{-m}$. Let $x \in \Sigma$ be such that $d(x, \mathbb{K}) = 2^{-m}$, and pick $x_K \in \mathbb{K}$ such that x and x_K coincide for exactly m initial digits. Due to the definition of $\mathbb{K}$, $m \geqslant 2$ (for any x) but we assume in the following that $m \geqslant 3$. By (6) we have $d(H^n x, \mathbb{K}) = d(H^n x, H^n x_K) = 2^{-2^n m}$. Assume that the first digit of x is 0. Then $H^n(x)$ coincides with ρ_0 at least for 2^n digits.

Assume now that $H^n x$ has an accident at the j-th shift, $1 \leqslant j < 2^n$, so there is $y \in \mathbb{K}$ such that $d(\sigma^j \circ H_n(x), y) < 2d(\sigma^j \circ H_n(x), \sigma^j \circ H_n(x_K))$.

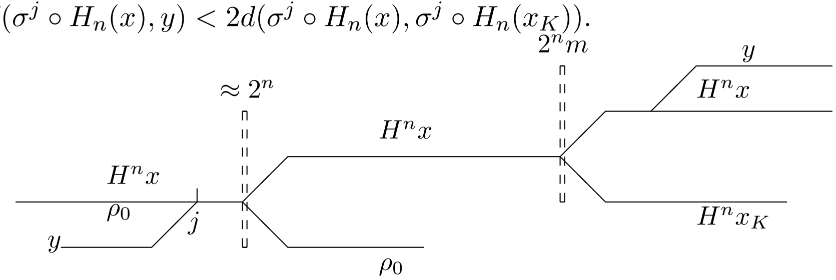


FIGURE 2. Half of the sum $\mathcal{R}^n V$ can easily be estimated.

The last point in Proposition 3 shows $j \geqslant 2^{n-1}$. Therefore, using again that the sum approximates the Riemann integral,

$$(\mathcal{R}^n V_0)(x) \geqslant \frac{1}{2^n} \sum_{j=0}^{(2^n/2)-1} \frac{1}{m - j/2^n} \to_{n\to\infty} \int_0^{\frac{1}{2}} \frac{1}{m-x}\, dx \geqslant \frac{1}{2m}.$$

The worst case scenario for the upper bound is when there is no accident, and then

$$(\mathcal{R}^n V_0)(x) = \sum_{j=0}^{2^n-1} \frac{1}{2^n m - j} \to_{n\to\infty} \int_0^1 \frac{1}{m-x}\, dx \leqslant \frac{1}{m-1} \tag{8}$$

as required.

Remark 3. *Note that the largest distance between $\mathbb{K}$ and points $\sigma^k(H^n(x))$ with $k \in [\![0, 2^n - 1]\!]$ is smaller than $2^{-(2^n m - 2^n + 1)} \leqslant 2^{-2^n}$. This largest distance thus tends to 0 super-exponentially fast as $n \to \infty$.* ■

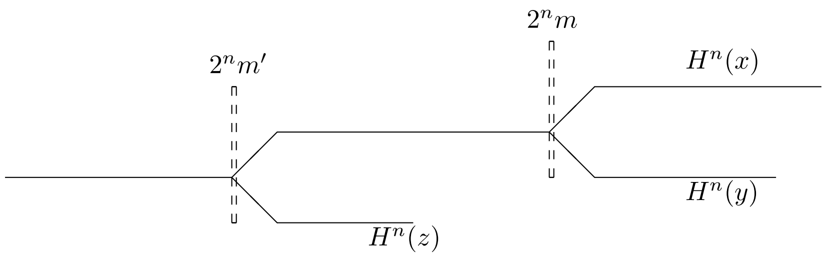
Step 3. We prove here equicontinuity for $\mathcal{R}^n(V_0)$. Namely, there exists some positive κ, such that for every n, for every x and y

$$|\mathcal{R}^n(V_0)(x) - \mathcal{R}^n(V_0)(y)| \leqslant \frac{\kappa}{|\log_2 d(x,y)|},$$

holds.

Assume that x and $y \in \Sigma$ coincide for m digits. We consider two cases.

Case 1: $d(x, \mathbb{K}) = 2^{-m'} =: d(x,z)$, with $m' < m$ (and $z \in \mathbb{K}$).



If there are no accidents for $\sigma^j \circ H^n(x)$ for $j \in [\![0, 2^n[\![$, then for every j,

$$d(\sigma^j(H^n(x)), \mathbb{K}) = d(\sigma^j(H^n(y)), \mathbb{K}) = d(\sigma^j(H^n(x)), \sigma^j(H^n(z))),$$

and $V_0(\sigma^j(H^n(x))) = V_0(\sigma^j(H^n(y)))$. This yields $(\mathcal{R}^n V_0)(x) = (\mathcal{R}^n V_0)(y)$.

Case 2: If there is an accident, say at time j_0, then two sub-cases can happen.

Subcase 2-1. The accident is due to a point z' that separates before $2^n m$, see Figure 3.

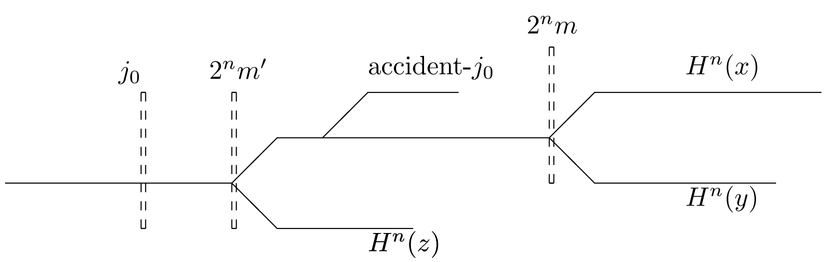


FIGURE 3. Comparing sequence when the accident occurs before separation.

Again, we claim that $V_0(\sigma^j(H^n(x))) = V_0(\sigma^j(H^n(y)))$ holds for $j \leqslant j_0 - 1$, but also for $j \geqslant j_0$ but smaller than the (potential) second accident. Going further, we refer to cases 2-2 or 1.

Sub-case 2-2. The accident is due to a point much closer to $H^n(x)$ than to $H^n(y)$, see Figure 4.

In that case we recall that the first accident cannot happen before 2^{n-1}, hence $j_0 \geqslant 2^{n-1}$. Again, for $j \leqslant j_0 - 1$ we get $V_0(\sigma^j(H^n(x))) = V_0(\sigma^j(H^n(y)))$. By definition of accident we get

$$\max\left\{V_0(\sigma^{j+2^{n-1}}(H^n(x)))\ ,\ V_0(\sigma^{j+2^{n-1}}(H^n(y)))\right\} \leqslant \frac{1}{2^n m - 2^{n-1} - j}$$

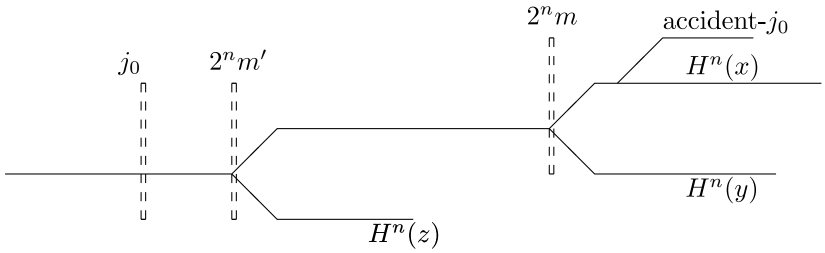


FIGURE 4. Comparing sequence when the accident occurs after separation.

for $j \geqslant j_0$. This yields

$$|(\mathcal{R}^n V_0)(x) - (\mathcal{R}^n V_0)(y)| \leqslant \sum_{k=j}^{2^{n-1}} \frac{2}{2^n m - 2^{n-1} - j} = \frac{1}{2^n} \sum_{k=j}^{2^{n-1}} \frac{2}{m - \frac{1}{2} - \frac{j}{2^n}}.$$

This last sum is a Riemann sum and is thus (uniformly in n) comparable to the associated integral $\int_0^{\frac{1}{2}} \frac{1}{m-\frac{1}{2}-t} dt \leqslant \frac{1}{2(m-1)}$.

Step 4. Following Step 3, the family $(\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{R}^k(V_0))_n$ is equicontinuous (and bounded), hence there exists accumulation points. Let us prove that $(\displaystyle\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{R}^k(V_0))_n$ actually converges.

Assume that $\widetilde{V}_1$ and $\widetilde{V}_2$ are two accumulation points. Note that both $\widetilde{V}_1$ and $\widetilde{V}_2$ are fixed points for $\mathcal{R}$. They are continuous functions and Steps 1 and 2 show that they satisfy

$$\frac{1}{2m} \leqslant \widetilde{V}_i(x) \leqslant \frac{1}{m} + o(\frac{1}{m}),$$

if $d(x, \mathbb{K}) = 2^{-m}$. From this we get

$$\widetilde{V}_1(x) - \widetilde{V}_2(x) \leqslant \frac{1}{2m} + o(\frac{1}{m}) = \frac{1}{2} V_0(x) + o(V_0(x)),$$

and then for every n,

$$\widetilde{V}_1 - \widetilde{V}_2 = \frac{1}{n}\sum_{k=0}^{n-1} \mathcal{R}^k(\widetilde{V}_1) - \mathcal{R}^k(\widetilde{V}_2) \leqslant \frac{1}{2}\frac{1}{n}\sum_{k=0}^{n-1} \mathcal{R}^k(V_0) + o(\mathcal{R}^k(V_0)).$$

We recall from Remark 2 that $o(\mathcal{R}^k(V_0))$ goes to 0 as $k \to \infty$. Taking the limit on the right hand side along the subsequence which converges to $\widetilde{V}_2$ we get

$$\widetilde{V}_1 - \widetilde{V}_2 \leqslant \frac{1}{2}\widetilde{V}_2 + o(V_0),$$

which is equivalent to $\frac{2}{3}\widetilde{V}_1 \leqslant \widetilde{V}_2 + o(V_0)$. Exchanging $\widetilde{V}_1$ and $\widetilde{V}_2$ we also get $\frac{2}{3}\widetilde{V}_2 \leqslant \widetilde{V}_1 + o(V_0)$. These two inequalities yield

$$\widetilde{V}_1 - \widetilde{V}_2 \leqslant \frac{1}{3} V_0 + o(V_0) \quad \text{and} \quad \widetilde{V}_2 - \widetilde{V}_1 \leqslant \frac{1}{3} V_0 + o(V_0).$$

Again, applying $\mathcal{R}^k$ on these inequalities and the Cesaro mean, we get

$$\widetilde{V}_1 - \widetilde{V}_2 \leqslant \frac{1}{3}\widetilde{V}_2 + o(V_0) \quad \text{and} \quad \widetilde{V}_2 - \widetilde{V}_1 \leqslant \frac{1}{3}\widetilde{V}_1 + o(V_0).$$

Iterating this process, we get that for every integer p,

$$\frac{p}{p+1}\widetilde{V}_2 + o(V_0) \leqslant \widetilde{V}_1 \leqslant \frac{p+1}{p}\widetilde{V}_2 + o(V_0).$$

This proves $\widetilde{V}_1 - \widetilde{V}_2 = o(V_0)$, *i.e.*, $(\widetilde{V}_1 - \widetilde{V}_2)(x) \to 0$ faster than $V_0(x)$ as $x \to \mathbb{K}$ (see again Remark 2). But $\widetilde{V}_1 - \widetilde{V}_2$ is also fixed by $\mathcal{R}$, so we can apply (8) with a factor $o(V_0)$ in front. This shows that $\widetilde{V}_1 = \widetilde{V}_2$, and hence the convergence of the Cesaro mean $(\frac{1}{n}\sum_{k=0}^{n-1}\mathcal{R}^k(V_0))_n$. This finishes the proof of Theorem 2. □

2.4. **More results on fixed points of $\mathcal{R}$.** The same proof also proves a more general result:

Proposition 9. *Let a be a real positive number. Take $V(x) = \frac{1}{n^a} + o(\frac{1}{n^a})$ if $d(x,\mathbb{K}) = 2^{-n}$. Then, for $a > 1$, $\lim_{n\to\infty}\mathcal{R}^nV \equiv 0$ and for $a < 1$, $\lim_{n\to\infty}\mathcal{R}^nV \equiv \infty$.*

Proof. Immediate, since the Riemann sum as in (8) has a factor $2^{n(1-a)}$ in front of it. □

Consequently, any V satisfying $V(x) = \frac{1}{n} + o(\frac{1}{n})$ for $d(x,\mathbb{K}) = 2^{-n}$ belongs to the weak stable set $V \in \mathcal{W}^s(\widetilde{V})$ of the fixed potential $\widetilde{V}$ from Theorem 2. However, $\mathcal{W}^s(\widetilde{V})$ is in fact much larger:

Proposition 10. *If $V(x) = \frac{1}{n}g(x)$ for $d(x,\mathbb{K}) = 2^{-n}$ and $g : \Sigma \to \mathbb{R}$ a continuous function, then $\frac{1}{j}\sum_{k=0}^{j-1}\mathcal{R}^k(V) \to \widetilde{V}\cdot\int_{\mathbb{K}} g\ d\mu_{\mathbb{K}}$.*

Proof. Take $\varepsilon > 0$ arbitrary, and take $r \in \mathbb{N}$ so large that $\sup|g|2^{-r} \leqslant \varepsilon$ and if $d(x,\mathbb{K}) = d(x,x_{\mathbb{K}}) \leqslant 2^{-r}$, then $|g(x) - g(x_{\mathbb{K}})| \leqslant \varepsilon$. Next take $k \in \mathbb{N}$ so large that if $k = r+s$, then

$$\left|\frac{1}{2^s}\sum_{i=0}^{2^s-1} g(\sigma^i(y)) - \int g\ d\mu_{\mathbb{K}}\right| \leqslant \varepsilon.$$

uniformly over $y \in \mathbb{K}$. Then we can estimate

$$\begin{aligned}
(\mathcal{R}^kV)(x) &= \sum_{j=0}^{2^k-1} V\circ\sigma^j\circ H^k(x) \\
&\leqslant \frac{1}{2^k}\sum_{j=0}^{2^k-1}\frac{1}{m-\frac{j}{2^k}}g\circ\sigma^j\circ H^k(x) \\
&= \frac{1}{2^r}\sum_{t=0}^{2^r-1}\frac{1}{2^s}\sum_{i=0}^{2^s-1}\frac{1}{m-\frac{1}{2^k}(2^st+i)}g\circ\sigma^{2^st+i}\circ H^k(x) \\
&= \frac{1}{2^r}\sum_{t=0}^{2^r-2}\frac{1}{2^s}\sum_{i=0}^{2^s-1}\left(\frac{1}{m-\frac{t}{2^r}} + O(2^{-r})\right)\cdot\int_{\mathbb{K}}(g\ d\mu_{\mathbb{K}} + O(\varepsilon)) \\
&\quad + \frac{1}{2^r}\frac{1}{2^s}\sum_{j=0}^{2^s-1}\frac{1}{m-\frac{1}{2^k}(2^k-2^s+j)}\sup|g| \\
&\to \int_0^1\frac{1}{m-x}dx\cdot\int_{\mathbb{K}} g\ d\mu_{\mathbb{K}} + O(3\varepsilon).
\end{aligned}$$

Since ε is arbitrary, we find $\limsup_k(\mathcal{R}^k V)(x) \leqslant \frac{1}{m}\cdot\int_{\mathbb{K}} g\ d\mu_{\mathbb{K}} + o(\frac{1}{m})$. Similar to Step 2 in the proof of Theorem 2, we find $\limsup_k(\mathcal{R}^k V)(x) \geqslant \frac{1}{2m}\cdot\int_{\mathbb{K}} g\ d\mu_{\mathbb{K}} + o(\frac{1}{m})$. From this, using the argument of Step 3 in the proof of Theorem 2, we conclude that for the Cesaro means, $\lim_n \frac{1}{n}\sum_{k=0}^{n-1}(\mathcal{R}^k V)(x) = \widetilde{V}(x)\cdot\int_{\mathbb{K}} g\ d\mu_{\mathbb{K}}$. □

2.5. **Unbounded fixed points of $\mathcal{R}$.** The application to Feigenbaum maps discussed in the Appendix of this paper suggests the existence of unbounded fixed points V_u of $\mathcal{R}$ as well. They can actually be constructed explicitly using the disjoint decomposition

$$\Sigma \setminus \sigma^{-1}\{\rho_0,\rho_1\} = \sqcup_{k\geqslant 0}\left((\sigma\circ H)^k(\Sigma)\setminus(\sigma\circ H)^{k+1}(\Sigma)\right).$$

If we set

$$V_u|_{H(\Sigma)} = g \quad \text{and} \quad V_u(x) = V_u(y) - V_u\circ H(y) \quad \text{for } x = \sigma\circ H(y), \tag{9}$$

then V_u is well-defined and $\mathcal{R}V_u = V_u$ on $\Sigma\setminus\sigma^{-1}\{\rho_0,\rho_1\}$. The simplest example is

$$V_u|_{(\sigma\circ H)^k(\Sigma)\setminus(\sigma\circ H^{k+1})(\Sigma)} = (1-k)\alpha, \tag{10}$$

and we will explore this further for phase transitions in Section 3.

For $x\in\Sigma\setminus\sigma\circ H(\Sigma)$ and $x^k = (\sigma\circ H)^k(x)$, we have

$$V_u(x^k) = g(x) - \sum_{j=1}^{k} g\circ\sigma^{2^j-2}\circ H^j(x).$$

Now for $x\in[1]$

$$\sigma^{2^j-2}\circ H^j(x) \to \begin{cases} \sigma^{-2}(\rho_0) & \text{along odd } j\text{'s},\\ \sigma^{-2}(\rho_1) & \text{along even } j\text{'s},\end{cases}$$

and the reverse formula holds for $x\in[0]$. In either case, $V(x^k)\sim\frac{k}{2}[g\circ\sigma^{-2}(\rho_0)+g\circ\sigma^{-2}(\rho_1)]$. Therefore, unless $g\circ\sigma^{-2}(\rho_0)+g\circ\sigma^{-2}(\rho_1) = 0$, the potential V_u is unbounded near $\lim_{k\to\infty}(\sigma\circ H)^k(x) = \{\sigma^{-1}(\rho_0)\ ,\ \sigma^{-1}(\rho_1)\}$, cf. Lemma 6.

Remark 4. *A variation of this stems from the decomposition*

$$\Sigma\setminus\{\rho_0,\rho_1\} = \sqcup_{k\geqslant 0}\left(H^k(\Sigma)\setminus H^{k+1}(\Sigma)\right).$$

In this case, if we define

$$V'_u|_{\sigma\circ H(\Sigma)} = g \quad \textit{and} \quad V'_u(x) = V'_u(y) - V'_u\circ\sigma(x) \quad \textit{for } x = H(y),$$

then $V'_u = \mathcal{R}V'_u$ *on* $\Sigma\setminus\{\rho_0,\rho_1\}$. ■

3. Thermodynamic formalism

In this section we prove Theorems 4, 5 and 6. In the first subsection we define an induced transfer operator as in [19] and use its properties. Then we prove both theorems.

3.1. **General results and a key proposition.** Let $V : \Sigma \to \mathbb{R}$ be some potential function, and let J be any cylinder such that on it, the distance to $\mathbb{K}$ is constant, say δ_J. Consider the first return map $T : J \to J$, say with return time $\tau(x) = \min\{n \geqslant 1 : \sigma^n(x) \in J\}$, so $T(x) = \sigma^{\tau(x)}(x)$. The sequence of successive return times is then denoted by $\tau^k(x)$, $k = 1, 2, \dots$ The transfer operator is defined as

$$(\mathcal{L}_{z,\gamma}g)(x) = \sum_{T(y)=x} e^{\Phi_{z,\gamma}(y)} g(y) \tag{11}$$

where $\Phi_{z,\gamma}(y) := -\gamma(S_nV)(y) - nz$ if $\tau(y) = n$. For a given test function g and a point $x \in J$, $(\mathcal{L}_{z,\gamma}g)(x)$ is thus a power series in e^{-z}.

These operators extend the usual transfer operator. They were introduced in [19] and allow us to define *local equilibrium states*, *i.e.,* equilibrium states for the potentials of the form $\Phi_{z,\gamma}$ and the dynamical system (J, T). These local equilibrium states are later denoted by $\nu_{z,\gamma}$.

We emphasize that, using induction on J, these operators $\mathcal{L}_{z,\gamma}$ allow us to construct equilibrium states for potentials which do not necessarily satisfy the Bowen condition (such as *e.g.* the Hofbauer potential).

Nevertheless, we need the following *local Bowen condition*: there exists C_V (possibly depending on J) such that

$$|(S_nV)(x) - (S_nV)(y)| \leqslant C_V, \tag{12}$$

whenever $x, y \in J$ coincide for $n := \tau^k(x) = \tau^k(y)$ indices. This holds, *e.g.* if $V(x)$ depends only on the distance between x and $\mathbb{K}$.

Lemma 11. *Let $x \in J$ and let γ and z be such that $(\mathcal{L}_{z,\gamma}1\!\!1_J)(x) < \infty$. Then $(\mathcal{L}_{z,\gamma}g)(y) < \infty$ for every $y \in J$ and for every continuous function $g : J \to \mathbb{R}$.*

Proof. Note that for any $x, y \in J$, $(\mathcal{L}_{z,\gamma}1\!\!1_J)(x) \approx e^{\pm C_V}(\mathcal{L}_{z,\gamma}1\!\!1_J)(y)$. Indeed, if x' and y' are two preimages of x and y in J, with the same return time n and such that for every $k \in [\![0, n]\!]$ $\sigma^k(x')$ and $\sigma^k(y')$ are in the same cylinder, then

$$|(S_nV)(x') - (S_nV)(y')| \leqslant C_V.$$

Recall that J is compact, and that every continuous function g on J is bounded. Hence convergence (*i.e.,* as power series) of $(\mathcal{L}_{z,\gamma}1\!\!1_J)(x)$ ensures uniform convergence over $y \in J$ for any continuous g. This finishes the proof of the lemma. □

For fixed γ, there is a critical z_c such that $(\mathcal{L}_{z,\gamma}1\!\!1_J)(x)$ converges for all $z > z_c$ and z_c is the smallest real number with this property. Lemma 11 shows that z_c is independent of x. The next result is straightforward.

Lemma 12. *The spectral radius $\lambda_{z,\gamma}$, of $\mathcal{L}_{z,\gamma}$ is decreasing in both γ and z.*

We are interested in the critical z_c and the pressure $\mathcal{P}(\gamma)$, both as function of γ. Both curves are decreasing (or at least non-increasing). If the curve $\gamma \mapsto z_c(\gamma)$ avoids the horizontal axis, then there is no phase transition:

Proposition 13. *Let V be continuous and satisfying the local Bowen condition* (12) *for every cylinder J disjoint and at constant distance from $\mathbb{K}$. Then the following hold:*

1. *For every $\gamma \geqslant 0$, the critical $z_c(\gamma) \leqslant \mathcal{P}(\gamma)$.*
2. *Assume that the pressure $\mathcal{P}(\gamma) > -\gamma \int V\, d\mu_{\mathbb{K}}$. Then there exists a unique equilibrium state for $-\gamma V$ and it gives a positive mass to every open set in Σ. Moreover $z_c(\gamma) < \mathcal{P}(\gamma)$ and $\mathcal{P}(\gamma)$ is analytic on the largest open interval where the assumption holds.*

3 *If $(\mathcal{L}_{z,\gamma} 1\!\!1_J)(\xi)$ diverges for every (or some) ξ and for $z = z_c(\gamma)$, then $\mathcal{P}(\gamma) > z_c(\gamma)$ and there is a unique equilibrium state for $-\gamma V$.*

Proof. There necessarily exists an equilibrium state for $-\gamma V$. Indeed, the potential is continuous and the metric entropy is upper semi-continuous. Therefore any accumulation point as $\varepsilon \to 0$ of a family of measures ν_ε satisfying

$$h_{\nu_\varepsilon}(\sigma) - \gamma \int V\, d\nu_\varepsilon \geqslant \mathcal{P}(\gamma)$$

is an equilibrium state.

The main argument in the study of local equilibrium states as in [19] is that $z > z_c(\gamma)$ (to make the transfer operator "converges") and that V satisfies the local Bowen property (12). This property is used in several places and in particular, it yields for every x and y in J and for every n:

$$e^{-\gamma C_V} \leqslant \frac{(\mathcal{L}^n_{z\gamma} 1\!\!1_J)(x)}{(\mathcal{L}^n_{z\gamma} 1\!\!1_J)(y)} \leqslant e^{\gamma C_V}.$$

To prove part 1., recall that

$$(\mathcal{L}_{z,\gamma} 1\!\!1_J)(x) := \sum_{n=1}^{\infty} \left(\sum_{x', T(x')=x, \tau(x)=n} e^{-\gamma (S_n V)(x')} \right) e^{-nz},$$

which yields that $z_c = \limsup_n \frac{1}{n} \log \left(\sum_{x', T(x')=x, \tau(x)=n} e^{-\gamma(S_n V)(x')} \right)$. To prove the inequality $z_c(\gamma) \leqslant \mathcal{P}(\gamma)$, we copy the proof of Proposition 3.10 in [20]. Define the measure $\widetilde{\nu}$ as follows: for x in J and for each T-preimage y of x there exists a unique $\tau(y)$-periodic point $\xi(y) \in J$, coinciding with y until $\tau(y)$. Next we define the measure $\widetilde{\nu}_n$ as the probability measure proportional to

$$\sum_{\xi(y), \tau(y)=n} e^{\Phi_{\mathcal{P}(\gamma),\gamma}(\xi(y))} \left(\sum_{j=0}^{n-1} \delta_{\sigma^j \xi(y)} \right) = \sum_{\xi(y), \tau(y)=n} e^{-\gamma (S_n V)(\xi(y)) - n\mathcal{P}(\gamma)} \left(\sum_{j=0}^{n-1} \delta_{\sigma^j \xi(y)} \right).$$

The measure $\widetilde{\nu}$ is an accumulation point of $(\widetilde{\nu}_n)_{n\in\mathbb{N}}$. It follows from the proof of [18, Lemma 20.2.3, page 264] that

$$z_c(\gamma) \leqslant h_{\widetilde{\nu}}(\sigma) - \gamma \int V\, d\widetilde{\nu} \leqslant \mathcal{P}(\gamma). \tag{13}$$

Remark 5. *We emphasize that $\widetilde{\nu}_n(J) = \frac{1}{n}$ for each n, which shows that $\widetilde{\nu}(J) = 0$.* ■

Now we prove part 2. Let μ_γ be an ergodic equilibrium state for $-\gamma V$. The assumption $\mathcal{P}(\gamma) > -\gamma \int V\, d\mu_{\mathbb{K}}$ means that the unique shift-invariant measure on $\mathbb{K}$ cannot be an equilibrium state (since $\sigma|_{\mathbb{K}}$ has zero entropy). Hence μ_γ gives positive mass to some cylinder J in $\mathbb{K}^c$. Thus the conditional measure

$$\nu_\gamma(\cdot) := \mu_\gamma(\cdot \cap J)/\mu_\gamma(J). \tag{14}$$

is T-invariant (using the above notations).

We now focus on the convergence (as power series) of $(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(x)$ for any $x \in J$ and $z = \mathcal{P}(\gamma)$. The inequality $z_c(\gamma) \leqslant \mathcal{P}(\gamma)$ does not ensure convergence of $(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(x)$ for $z = \mathcal{P}(\gamma)$. Again, we copy and adapt arguments from [20, Proposition 3.10] to get that $(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(x)$ converges and that the $\Phi_{z,\gamma}$-pressure is non-positive for $z = \mathcal{P}(\gamma)$.

In the case $z > \mathcal{P}(\gamma)$, so $z > z_c(\gamma)$, we can apply the local thermodynamic formalism for $\Phi_{z,\gamma}$. Moreover $z > z_c(\gamma)$ means that $\frac{\partial}{\partial z}(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(x)$ converges. This implies by [19, Proposition 6.8] that there exists a unique equilibrium state $\nu_{z,\gamma}$ on J for T and for the potential $\Phi_{z,\gamma}$, and that the expectation $\int_J \tau \, d\nu_{z,\gamma} < \infty$. In other words, there exists a shift-invariant probability measure $\mu_{z,\gamma}$ such that

$$\mu_{z,\gamma}(J) > 0, \text{ and } \nu_{z,\gamma}(\cdot) := \frac{\mu_{z,\gamma}(\cdot \cap J)}{\mu_{z,\gamma}(J)}.$$

The equality $h_{\nu_{z,\gamma}}(T) + \int \Phi_{z,\gamma} \, d\nu_{z,\gamma} = \log \lambda_{z,\gamma}$ (the spectral radius for $\mathcal{L}_{z,\gamma}$) shows that

$$h_{\mu_{z,\gamma}}(\sigma) - \gamma \int V \, d\mu_{z,\gamma} = z + \mu_{z,\gamma}(J) \log \lambda_{z,\gamma}.$$

As $z > \mathcal{P}(\gamma)$ we get $\lambda_{z,\gamma} \leqslant 1$. Now the Bowen property of the potential shows that for every $x \in J$ and for every $n \geqslant 1$:

$$(\mathcal{L}^n_{z,\gamma}\mathbb{1}_J)(x) = e^{\gamma C_V} \lambda^n_{z,\gamma}.$$

The power series is decreasing in z, thus the monotone Lebesgue convergence theorem shows that it converges for $z = \mathcal{P}(\gamma)$. For this value of the parameter z, the spectral radius $\lambda_{\mathcal{P}(\gamma),\gamma} \leqslant 1$. Following [19], there exists a unique local equilibrium state, $\nu_{\mathcal{P}(\gamma),\gamma}$ with pressure $\log \lambda_{\mathcal{P}(\gamma),\gamma} \leqslant 1$. This proves that the $\Phi_{\mathcal{P}(\gamma),\gamma}$-pressure is non-positive.

Now, we prove that the $\Phi_{z,\gamma}$-pressure is non-negative for $z = \mathcal{P}(\gamma)$. Indeed, by Abramov's formula

$$\begin{aligned} 0 &= h_{\mu_\gamma}(\sigma) - \gamma \int V \, d\mu_\gamma - \mathcal{P}(\gamma) \\ &= \mu_\gamma(J) \left(h_{\nu_\gamma}(T) - \gamma \int (S_{\tau(x)}V)(x) \, d\nu_\gamma(x) - \mathcal{P}(\gamma) \int \tau \, d\nu_\gamma \right), \end{aligned}$$

which yields

$$h_{\nu_\gamma}(T) - \gamma \int S_{\tau(x)}(V)(x) - \mathcal{P}(\gamma)\tau(x) \, d\nu_\gamma(x) = 0.$$

Finally, as the $\Phi_{\mathcal{P}(\gamma),\gamma}$-pressure is non-negative and non-positive, it is equal to 0. It also has a unique equilibrium state which is a Gibbs measure (in J and for the first-return map T). As the conditional measure ν_γ has zero $\Phi_{\mathcal{P}(\gamma),\gamma}$-pressure, it is the unique local equilibrium state.

The local Gibbs property proves that ν_γ gives positive mass to every open set in J, and the mixing property shows that the global shift-invariant measure μ_γ gives positive mass to every open set in Σ. We can thus copy the argument to show it is uniquely determined on each cylinder which does not intersect $\mathbb{K}$ (here we use the assumption that the potential satisfies (12) for each cylinder J with empty intersection with $\mathbb{K}$).

It now remains to prove analyticity of the pressure. Equality (13) gives $z_c(\gamma) \leqslant h_{\widetilde{\nu}}(\sigma) - \gamma \int V\, d\widetilde{\nu}$. Remark 5 states that $\widetilde{\nu}(J) = 0$, and uniqueness of the equilibrium state shows that $\widetilde{\nu}$ cannot be this equilibrium state (otherwise we would have $\widetilde{\nu}(J) > 0$). Hence, $z_c(\gamma)$ is strictly less than $\mathcal{P}(\gamma)$. Then, we use [16] to get analyticity in each variable z and γ, and the analytic version of the implicit function theorem (see [25]) shows that $\mathcal{P}(\gamma)$ is analytic.

The proof of part 3 is easier. The divergence of $\mathcal{L}_{z,\gamma}(\mathbb{1}_J)(\xi)$ for some ξ and $z = z_c(\gamma)$ ensures the divergence for every ξ, and then Lemma 12 and the local Bowen condition show that $\lambda_{z,\gamma}$ goes to ∞ as z goes to $z_c(\gamma)$. This means that there exists a unique $Z > z_c(\gamma)$ such that $\lambda_{Z,\gamma} = 1$. Using the work done in the proof of point 2, we let the reader check that necessarily $Z = \mathcal{P}(\gamma)$ and the local equilibrium state produces a global equilibrium state (see also [19]).

This finishes the proof of the proposition. □

Actually, Proposition 13 says a little bit more. If the second assumption is satisfied, namely $\mathcal{P}(\gamma) > -\int V\, d\mu_{\mathbb{K}}$, then the unique equilibrium state for V in Σ is the measure obtained (using Equation (14)) from the unique equilibrium state $\nu_{\mathcal{P}(\gamma),\gamma}$ for the dynamical system (J, T) and associated to the potential $\Phi_{\mathcal{P}(\gamma),\gamma}$. Therefore, two special curves z as function of γ appear, see Figure 5. The first is $z_c(\gamma)$, and the second is $\mathcal{P}(\gamma)$, defined by the implicit equality

$$\log \lambda_{\mathcal{P}(\gamma),\gamma} = 0.$$

We claim that these curves are convex.

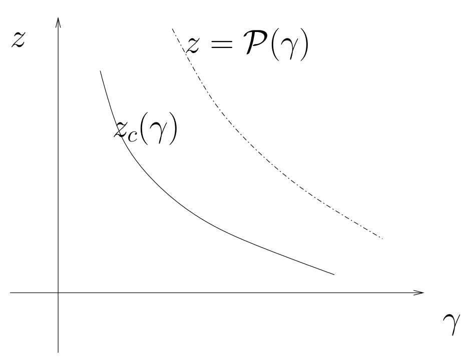


FIGURE 5. Two important values of z as function of γ.

3.2. **Counting excursions close to $\mathbb{K}$.** Let $x \in \Sigma$ and $n \in \mathbb{N}$ be such that for $k \in [\![0, n-1]\!]$, $d(\sigma^k(x), \mathbb{K}) \leqslant 2^{-5}\delta_J$. We divide the piece of orbit $x, \sigma(x), \dots, \sigma^{n-1}(x)$ into pieces between accidents. We take $b_0 = 0$ by default, and let $y^0 \in \mathbb{K}$ be the point closest to x. Inductively, set

$$\begin{array}{rcl}
b_1 & = & \min\{j \geqslant 1 : d(\sigma^j(x), \mathbb{K}) \leqslant d(\sigma^j(x), \sigma^j(y^0))\}, \\
 & & \quad y^1 \in \mathbb{K} \text{ is point closest to } \sigma^{b_1}(x). \\
b_2 & = & \min\{j \geqslant 1 : d(\sigma^{j+b_1}(x), \mathbb{K}) \leqslant d(\sigma^{j+b_1}(x), \sigma^j(y^1))\}, \\
 & & \quad y^2 \in \mathbb{K} \text{ is point closest to } \sigma^{b_1+b_2}(x). \\
b_3 & = & \min\{j \geqslant 1 : d(\sigma^{j+b_1+b_2}(x), \mathbb{K}) \leqslant d(\sigma^{j+b_1+b_2}(x), \sigma^j(y^2)\}, \\
 & & \quad y^3 \in \mathbb{K} \text{ is point closest to } \sigma^{b_1+b_2+b_3}(x) \\
\vdots & \vdots & \vdots
\end{array}$$

and $d_j = -\log_2 d(\sigma^{\sum_{i<j} b_i}(x), \mathbb{K}) = -\log_2 d(\sigma^{\sum_{i<j} b_i}(x), y^{j-1})$ expresses how close the image of x is to $\mathbb{K}$ at the $j-1$-st accident.

Following Proposition 3, $d_j - b_j$ is of the form $3^{\varepsilon_j} 2^{k_j}$, with $\varepsilon_j \in \{0,1\}$ and $d_{j+1} > d_j - b_j$ by definition of an accident. One problem we will have to deal with, is to count the possible accidents during a very long piece of orbit: if we know $d_j - b_j$ can we determine the possible values of d_j? As it is stated in Subsection 2.1, accidents occur at bispecial words which have to be prefixes of $\tau_n\bar{\tau}_n\tau_n$ or $\bar{\tau}_n\tau_n\bar{\tau}_n$, and are words of the form τ_k, $\bar{\tau}_k$, $\tau_k\bar{\tau}_k\tau_k$ or $\bar{\tau}_k\tau_k\bar{\tau}_k$.

From now on, we pick some non-negative potential V and assume it satisfies hypotheses of Proposition 13. Namely, our potentials are of the form $V(x) = \frac{1}{n^a} + o(\frac{1}{n^a})$ if $\log_2(d(x, \mathbb{K})) = -n$. They satisfy hypotheses of Proposition 13, and furthermore, the Birkhoff sums are locally constant.

Moreover, for x and y in J coinciding until $n = \tau(x) = \tau(y)$, the assumption $d(\mathbb{K}, J) = \delta_J = d(x, \mathbb{K}) = d(y, \mathbb{K})$ shows that for every $j \leqslant n$,

$$d(\sigma^j(x), \mathbb{K}) = d(\sigma^j(y), \mathbb{K})$$

holds. Hence $\Phi_{\cdot,\gamma}$ satisfies the local Bowen property (12).

Let x be a point in J. We want to estimate $(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(x)$. Let y be a preimage of x for T. To estimate $\Phi(y)$, we decompose the orbit $y, \sigma(y), \ldots, \sigma^{\tau(y)-1}(y)$ where $\sigma^j(y)$ is reasonably far away from $\mathbb{K}$ (let $c_r \geqslant 0$ be the length of such piece) and *excursions* close to $\mathbb{K}$.

Definition 14. *An excursions begins at $\xi := \sigma^s(y)$ when ξ starts as ρ_0 or ρ_1 for at least $5 - \log_2 \delta_J$ digits (*i.e.*, $d(\xi', \rho_0)$ and $d(\xi', \rho_0) \leqslant \delta_J 2^{-5}$) and ends at $\xi' := \sigma^t(y)$ where $t > s$ is the minimal integer such that $d(\xi', \mathbb{K}) > \delta_J 2^{-5}$.*

If $\sigma^i(y)$ is very close to $\mathbb{K}$, then due to minimality of $(\mathbb{K}, \sigma)$ it takes a uniformly bounded (from above) number of iterates for an excursion to begin.

Note that each cylinder for the return map T is characterized by a *path*

$$c_0, \underbrace{b_{1,1}, b_{1,2}, \ldots, b_{1,N_1}}_{\text{first excursion}}, c_1, \underbrace{b_{2,1}, \ldots, b_{2,N_1}}_{\text{second excursion}}, c_2, \ldots, c_{M-1}, \underbrace{b_{M,1}, b_{M,2}, \ldots, b_{M,N_1}}_{M\text{-th excursion}}, c_M.$$

Any piece of orbit between two excursions or before the first excursion or after the last excursion is called a *free path*. Let s_r and t_r be the times where the r-th free path and

r-th excursion begin. Since J is disjoint from $\mathbb{K}$, each piece c_r of free path takes at least two iterates, so $c_r \geqslant 2$ for $0 \leqslant r \leqslant M$.

Due to the locally constant potential we are considering, $(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(x)$ is independent of the point x where it is evaluated. Hence, for the rest of the proofs in this section, unless it is necessary, we shall just write $\mathcal{L}_{z,\gamma}\mathbb{1}_J$. Our strategy is to glue on together paths in functions of their free-paths and the numbers of accidents during an excursion. This form clusters and the contribution of a cluster considering N accidents is of the form

$$E_{z,\gamma}(\mathbb{1}_J) := \sum_{N\geqslant 1} \underbrace{\sum_{\substack{\text{allowed}\\ (b_j)_{j=1}^N,\ (d_j)_{j=1}^N}} \exp\left(-\gamma \sum_{j=1}^N S_j V\right) \exp\left(-\sum_{j=1}^N b_j z\right) D_N}_{A_N}, \tag{15}$$

where S_jV is the Birkhoff sum of the potential after the j^{th} accident but before the next one and the quantity $D_N = e^{\varphi_N-(d_N-b_N)z}$ is the contribution of the last part of the orbit after the N^{th} accident. By definition of an accident, this contribution is larger than if there would be no accident. Therefore for non-negative z, $\frac{e^{-(d_N-b_N)z}}{(d_N-b_N)^\gamma} \leqslant D_N \leqslant 1$. The quantity A_N is the sum of the contribution of the cluster with N accident.

Thus we have

$$\begin{aligned}(\mathcal{L}_{0,\gamma}\mathbb{1}_J)(x) &= \left(\sum_{c_0\geqslant 5} \sum_{\substack{\text{free}\\ c_0-\text{paths}}} e^{-\gamma\sum_{n=0}^{c_0-1} V(\sigma^n(y))-c_0 z}\right) \times \\ &\left(\sum_{M\geqslant 0}\left[\left(\sum_{(c_r)_{r=1}^M}\sum_{\substack{\text{free}\\ c_r-\text{paths}}} e^{-\gamma\sum_{n=0}^{c_r-1} V(\sigma^{n+s_r}(y))-c_r z}\right)\times (E_{z,\gamma}(\mathbb{1}_J))^M\right]\right).\end{aligned} \tag{16}$$

3.3. **The potential n^{-a}: Proofs of Theorems 3 and 4 .**

3.3.1. *Proof of Theorem 3.* Here we deal with the case $a > 1$ and $V(x) = n^{-a}$ if $d(x,\mathbb{K}) = 2^{-n}$.

Proof of Theorem 3. Since $a > 1$,

$$\sum_{n=d-b+1}^{d} \frac{1}{n^a} \asymp \int_{d-b}^{d} \frac{1}{x^a}dx = \frac{1}{a-1}\left(\frac{1}{(d-b)^{a-1}} - \frac{1}{d^{a-1}}\right) \leqslant \frac{1}{a-1} < \infty,$$

for all values of $b < d$. To find a lower bound for $E_{z,\gamma}(\mathbb{1}_J)$ in (15), it suffices to take only excursions with a single accident, and sum over all possible d_1 with $d_1 - b_1 = 2^k$. Then

$$E_{z,\gamma}(\mathbb{1}_J) \geqslant \sum_k e^{-\gamma/(a-1)} = \infty,$$

regardless of the value of $\gamma > 0$.

By Proposition 13 (part 1) we get $\mathcal{P}(\gamma) > z_c(\gamma) \geqslant 0 = -\gamma \int V\, d\mu_{\mathbb{K}}$. Then Proposition 13 (part 3) ensures that there is no phase transition and that $\gamma \mapsto \mathcal{P}(\gamma)$ is positive and analytic on $[0, \infty)$.

To finish the proof of Theorem 3, we need to compute $\lim_{\gamma\to\infty} \mathcal{P}(\gamma)$. Let μ_γ be the unique equilibrium state for $-\gamma V$. Then

$$\frac{\mathcal{P}(\gamma)}{\gamma} = \frac{h_{\mu_\gamma}}{\gamma} - \int V\, d\mu_\gamma,$$

which yields $\limsup_{\gamma\to\infty} \frac{\mathcal{P}(\gamma)}{\gamma} \leqslant 0$, hence $\limsup_{\gamma\to\infty} \mathcal{P}(\gamma) \leqslant 0$. On the other hand, $\mathcal{P}(\gamma) \geqslant 0 = h_{\mu_{\mathbb{K}}} - \int V\, d\mu_{\mathbb{K}}$, hence $\liminf_{\gamma\to\infty} \mathcal{P}(\gamma) \geqslant 0$. □

3.3.2. *Proof of Theorem 4 for a special case.* Now take $a \in (0, 1)$ and

$$V(x) = n^{-a} \text{ if } d(x, \mathbb{K}) = 2^{-n},$$

so

$$\Phi_{z,\gamma}(x) = -\gamma S_n V(x) - nz = -\gamma \sum_{k=1}^{n} k^{-a} - nz.$$

The potential is locally constant on sufficiently small cylinder sets. It thus satisfies the local Bowen condition (12) and the hypotheses of Proposition 13 hold.

Recall that

$$\sum_{n=d-b+1}^{d} \frac{1}{n^a} \asymp \int_{d-b}^{d} \frac{1}{x^a} dx = \frac{1}{1-a}\left(d^{1-a} - (d-b)^{1-a}\right),$$

and we shall replace the discrete sum by the integral. The error involved in this can be incorporated in the changed coefficient $(1 \pm \varepsilon)\gamma$.

Our goal is to prove that $z_c(\gamma) = 0$ (for every γ) and that $\mathcal{L}_{0,\gamma}(\mathbb{1}_J)(x) \to 0$ as $\gamma \to \infty$ (for any $x \in J$). This will prove that there is a phase transition.

Lemma 15. *The series $(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(\xi)$ diverges for $z < 0$.*

Proof. We employ notations from (15) with our new V. In the full shift all orbits appear, and we are counting here only orbits which have only one excursion close to $\mathbb{K}$ without accident. For each j, we consider a piece of orbit of length $2^{k+1}(1 + 2j)$, coinciding with a piece of orbit within $\mathbb{K}$, and then "going back" to J. The quantity $E_{z,\gamma}(\mathbb{1}_J)$ is larger than the contribution of these excursions, which is

$$A_1^k(z) \geqslant \sum_{j=1}^{\infty} e^{-\frac{\gamma}{1-a}((2^{k+1}(1+2j))^{1-a}-1)-2^{k+1}(1+2j)z}.$$

As $a < 1$, $-2jz$ is eventually larger than $(2^{k+1}(1+2j))^{1-a}$ for $z < 0$ and the series trivially diverges. Then, $E_{z,\gamma}(\mathbb{1}_J)$ diverges as well, and (16) shows that $\mathcal{L}_{z,\gamma}(\mathbb{1}_J)$ diverges for every initial point $x \in J$. □

Let us now consider the case $z = 0$. As we are now looking for upper bounds, we can consider the b_j's and the d_j's as independent and sum over all possibilities (and thus forget the condition $d_{j+1} > d_j - b_j$). Note that we trivially have $D_N \leqslant 1$.

For a piece of orbit of length d and with an accident at b, $d-b=2^k$, the possible values of d's are among $2^k(1+\frac{j}{2})$, $j \geqslant 1$, and then $b = 2^{k-1}j$. If $d-b = 3\cdot 2^k$, then the possible values of d's are among $2^k(1+\frac{j}{2})$ with $j\geqslant 5$ and then $b=2^k(\frac{j}{2}-2)$.

Let
$$B(z) := \sum_{k=4}^{\infty}\sum_{j=1}^{\infty} e^{-\frac{\gamma}{1-a}\left(\left(2^k(1+\frac{j}{2})\right)^{1-a}-2^{k(1-a)}\right)-j2^{k-1}z}.$$
and
$$C(z) := \sum_{k=4}^{\infty}\sum_{j=5}^{\infty} e^{-\frac{\gamma}{1-a}\left(\left(2^k(1+\frac{j}{2})\right)^{1-a}-3^{1-a}2^{k(1-a)}\right)-2^{k-1}(j-4)z}.$$
The quantity $B(z)$ is an upper bound for the cluster with one excursion of the form $d-b=2^k$, and $C(z)$ is an upper bound for the cluster with one excursion of the form $d-b=3\cdot 2^k$.

Then multiplying N copies to estimate from above the contribution of excursion with N accidents we get
$$E_{z,\gamma}(1\!\!1_J) \leqslant \sum_N (B(z)+C(z))^N.$$
Hence
$$(\mathcal{L}_{0,\gamma}1\!\!1_J)(x) \;\leqslant\; \left(\sum_{c_0\geqslant 5}\;\sum_{\substack{\text{free}\\ c_0-\text{paths}}} e^{-\gamma\sum_{n=0}^{c_0-1}V(\sigma^n(y))-c_0z}\right)\times$$
$$\left(\sum_{M\geqslant 0}\left[\left(\sum_{(c_r)_{r=1}^M}\;\sum_{\substack{\text{free}\\ c_r-\text{paths}}} e^{-\gamma\sum_{n=0}^{c_r-1}V(\sigma^{n+s_r}(y))-c_rz}\right)\times\left(\sum_{N\geqslant 1}(B(z)+C(z))^N\right)^M\right]\right),$$
where the sum over $(c_r)_{r=1}^M$ is 1 by convention if $M=0$. The first factor (the sum over c_0) indicates the first cluster of free paths, and $c_0\geqslant 5$ by our choice of the distance δ_J.

Note that for the free pieces between excursions the orbit is relatively far from $\mathbb{K}$, so there is $\varepsilon>0$ depending only on δ_J such that
$$-c_r(\gamma+z) \leqslant \sum_{n=0}^{c_r-1} -\gamma V(\sigma^{n+s_r}(y)) - c_rz \leqslant -c_r(\varepsilon\gamma+z). \tag{17}$$

An upper bound for $\mathcal{L}_{z,\gamma}(1\!\!1_J)$ is obtained by taking an upper bound for B and C and majorizing the sum over the c_r free paths by taking the sum over all the c and the upper bound in (17).

Proof of Theorem 4. Lemma 15 shows that for every γ, $z_c(\gamma)\geqslant 0$. Our goal is to prove that $B(0)+C(0)$ can be made as small as wanted by choosing γ sufficiently large. This will imply that $z_c(\gamma)=0$ for sufficiently large γ and that the unique equilibrium state is $\mu_{\mathbb{K}}$. We compute $B(0)$ leaving the very similar computation for $C(0)$ to the reader.

Apply the inequality $1+u \geqslant 1+\log(1+u)$ for the value of u such that $1+u=(1+\frac{j}{2})^{1-a}$, to obtain $(1+\frac{j}{2})^{1-a}-1 \geqslant \log(1+\frac{j}{2})^{1-a}$, whence $e^{(1+\frac{j}{2})^{1-a}-1} \geqslant (1+\frac{j}{2})^{1-a}$. Raising this to the power $-\frac{\gamma}{1-a}2^{k(1-a)}$ and summing over j, we get

$$\begin{aligned}
\sum_{j=1}^{\infty} e^{-\frac{\gamma}{1-a}2^{k(1-a)}((1+\frac{j}{2})^{1-a}-1)} &\leqslant \sum_{j=1}^{\infty}(1+\frac{j}{2})^{-\gamma 2^{k(1-a)}} \\
&\leqslant \left(\frac{2}{3}\right)^{\gamma 2^{k(1-a)}} + \int_1^{\infty} \frac{dx}{(1+\frac{x}{2})^{-\gamma 2^{k(1-a)}}} \\
&= \left(1+\frac{3}{\gamma 2^{k(1-a)}-1}\right)\left(\frac{2}{3}\right)^{\gamma 2^{k(1-a)}}.
\end{aligned}$$

Therefore

$$B(0)=\sum_{k=4}^{\infty}\sum_{j=1}^{\infty} e^{\frac{\gamma}{1-a}2^{k(1-a)}(1-(1+\frac{j}{2})^{1-a})} \leqslant \sum_{k=4}^{\infty}\left(1+\frac{3}{\gamma 2^{k(1-a)}-1}\right)\left(\frac{2}{3}\right)^{\gamma 2^{k(1-a)}}$$

is clearly finite and tends to zero as $\gamma \to \infty$.

Now to estimate $(\mathcal{L}_{z,\gamma}1\!\!1_J)(x)$, we have to sum over the free periods as well and we have

$$\begin{aligned}
(\mathcal{L}_{0,\gamma}1\!\!1_J)(x) &\leqslant \left(\sum_{c\geqslant 5} 2^c e^{-\varepsilon\gamma c}\right)\cdot \sum_{M\geqslant 0}\left(\sum_{c\geqslant 1} 2^c e^{-c\varepsilon\gamma}(E_{0,\gamma}1\!\!1_J)(\xi)\right)^M \\
&\leqslant \frac{32e^{-5\varepsilon\gamma}}{1-2e^{-\varepsilon\gamma}} \sum_{M\geqslant 0}\left(\sum_{c\geqslant 1} 2^c e^{-c\varepsilon\gamma}\sum_{N\geqslant 1}(B(0)+C(0))^N\right)^M. \qquad (18)
\end{aligned}$$

The term in the brackets still tends to zero as $\gamma \to \infty$, and hence is less than 1 for $\gamma > \gamma_0$ and some sufficiently large γ_0. The double sum converges for such γ, so the critical $z_c(\gamma) \leqslant 0$ for $\gamma \geqslant \gamma_0$.

Lemma 15 shows that $z_c(\gamma)$ is always non-negative. Therefore $z_c(\gamma)=0$ for every $\gamma \geqslant \gamma_0$. In fact, for γ sufficiently large (and hence $e^{-5\varepsilon\gamma}$ is sufficiently small), the bound (18) is less than one: for every $x \in J$, $(\mathcal{L}_{0,\gamma}1\!\!1_J)(x) < 1$. This means that $\log \lambda_{0,\gamma}$, *i.e.*, the logarithm of the spectral radius of $\mathcal{L}_{z,\gamma}$, becomes zero at some value of γ, say γ_2.

Lemma 12 says that the spectral radius decreases in z. On the other hand the pressure $\mathcal{P}(\gamma)$ is non-negative because $\int V\, d\mu_{\mathbb{K}} = 0$. Moreover, the curve $z=\mathcal{P}(\gamma)$ is given by the implicit equality $\lambda_{\mathcal{P}(\gamma),\gamma}=1$. Therefore, the curve $\gamma \mapsto \mathcal{P}(\gamma)$ is below the curve $\gamma \mapsto \log \lambda_{0,\gamma}$. Thus it must intersect the horizontal axis at some $\gamma_1 \in [\gamma_0,\gamma_2]$ (see Figure 6).

For $\gamma > \gamma_1$ convexity yields $\mathcal{P}(\gamma)=0$, hence the function is not analytic at γ_1 and we have an ultimate phase transition (for $\gamma=\gamma_1$). Analyticity for $\gamma < \gamma_1$ follows from Proposition 13. □

3.3.3. *Proof of Theorem 4 for the general case.* Now we consider V such that

$$V(x)=n^{-a}+o(n^{-a}) \quad \text{if } d(x,\mathbb{K})=2^{-n}.$$

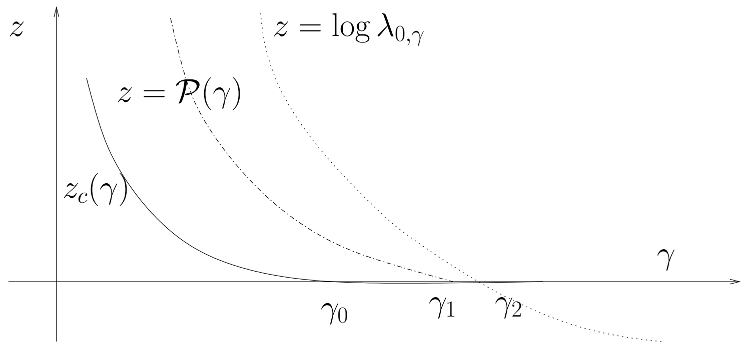


FIGURE 6. Phase transition at γ_1

For every fixed ε_0, there exists some N_0 such that for every $n \geqslant N_0$ and for x such that $d(x, \mathbb{K}) = 2^{-n}$,

$$\left| V(x) - \frac{1}{n^a} \right| \leqslant \frac{\varepsilon_0}{n^a}.$$

We can incorporate this perturbation in the free path, assuming that any path with length less than N_0 is a free path. Then all the above computations are valid provided we replace γ by $\gamma(1 \pm \varepsilon_0)$. This does not affect the results.

3.4. **The proof of Theorem 5.** As a direct application of Theorem 4, we can give a version of the Manneville-Pomeau map with a neutral Cantor set instead of a neutral fixed point.

Proof of Theorem 5. Pick $a > 0$, and consider V and γ_1 as in Subsection 3.3 (only for $a < 1$). For $a > 1$ we pick any positive γ_1. Define the canonical projection $\Pi : \Sigma \to [0, 1]$ by the dyadic expansion:

$$\Pi(x_0, x_1, x_2, \ldots) = \sum_j \frac{x_j}{2^{j+1}}.$$

It maps $\mathbb{K}$ to a Cantor subset of $[0, 1]$. Only dyadic points in $[0, 1]$ have two preimages under Π, namely $x_1 \ldots x_n 10^\infty$ and $x_1 \ldots x_n 01^\infty$ have the same image.

Lemma 16. *There exists a potential $W : \Sigma \to \mathbb{R}$ such that*

$$W(x) = \frac{1}{n^a} + o(\frac{1}{n^a}) \quad \text{if } d(x, \mathbb{K}) = 2^{-n},$$

and it is continuous at dyadic points:

$$W(x_1 \ldots x_n 10^\infty) = W(x_1 \ldots x_n 01^\infty),$$

and is positive everywhere except on $\mathbb{K}$ where it is zero.

Proof. Let us consider the multi-valued function $V \circ \Pi^{-1}$ on the interval. It is uniquely defined at each non-dyadic point. For a dyadic point, consider the two preimages $x_1 \ldots x_n 10^\infty$ and $x_1 \ldots x_n 01^\infty$ in Σ.

Case 1. The word $x_1 \dots x_n$ (which is $\mathbb{K}$-admissible) has a single suffix in $\mathbb{K}$, say 0. This means that $x_1 \dots x_n 0$ is an admissible word for $\mathbb{K}$ but not $x_1 \dots x_n 1$. Let $\underline{x}^- := x_1 \dots x_n 01^\infty$ and $\underline{x}^+ := x_1 \dots x_n 10^\infty$. Then

$$d(\underline{x}^+, \mathbb{K}) = 2^{-n} > d(\underline{x}^-, \mathbb{K}) > 2^{-n-4}, \tag{19}$$

where the last inequality comes from the fact that $x_1 \dots x_n 0111$ is not admissible for $\mathbb{K}$.

We modify the potential V on the right side hand of the dyadic point $\Pi(x_1 \dots x_n 10^\infty)$ as indicated on Figure 7.

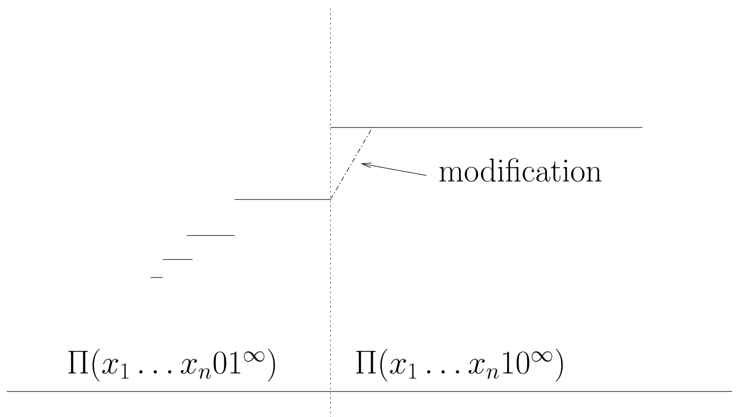


FIGURE 7. The modification for words with a single suffix

The inequalities of (19) yield

$$V(\underline{x}^-) = \frac{1}{(n+k)^a} = \frac{1}{n^a} - \frac{ak}{n^{a+1}} + o(\frac{1}{n^{a+2}}) = V(\underline{x}^+) + o(V(\underline{x}^+)),$$

where k is an integer in $[1, 4]$. As the modification is done "convexly", the new potential W satisfies for these modified points

$$W(x) = \frac{1}{n^a} + o(\frac{1}{n^a}) \quad \text{if } d(x, \mathbb{K}) = 2^{-n}.$$

Case 2. The word $x_1 \dots x_n$ (which is $\mathbb{K}$-admissible) has two suffixes in $\mathbb{K}$. It may be that $\underline{x}_+$ and $\underline{x}_-$ are at the same distance to $\mathbb{K}$ (see Figure 8). Then we do not need to change the potential around this dyadic point.

If $V(\underline{x}^+) \neq V(\underline{x}^-)$, neither $x_1 \dots x_n 0111$ nor $x_1 \dots x_n 1000$ are admissible for $\mathbb{K}$ and we modify the potential linearly in that region in the interval as Figure 9.

Again we have

$$V(\underline{x}^+) = \frac{1}{(n+j)^a} = \frac{1}{n^a} + o(\frac{1}{n^a}) \quad \text{and} \quad V(\underline{x}^-) = \frac{1}{(n+k)^a} = \frac{1}{n^a} + o(\frac{1}{n^a}),$$

where j and k are different integers in $\{1, 2, 3, 4\}$. Hence, for these points too, W satisfies

$$W(x) = \frac{1}{n^a} + o(\frac{1}{n^a}) \quad \text{if } d(x, \mathbb{K}) = 2^{-n}.$$

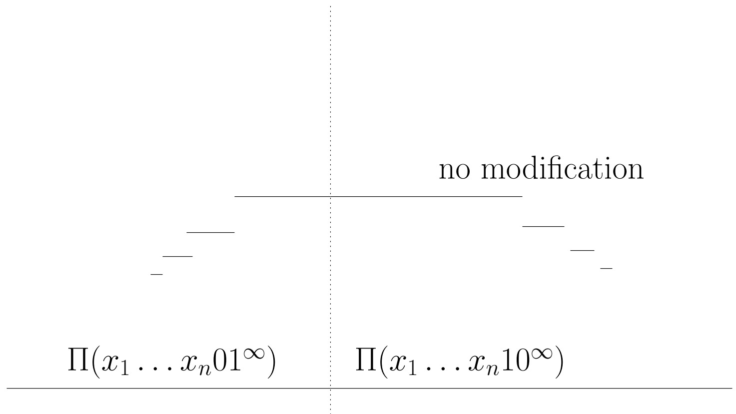


FIGURE 8. No modification with two different suffixes

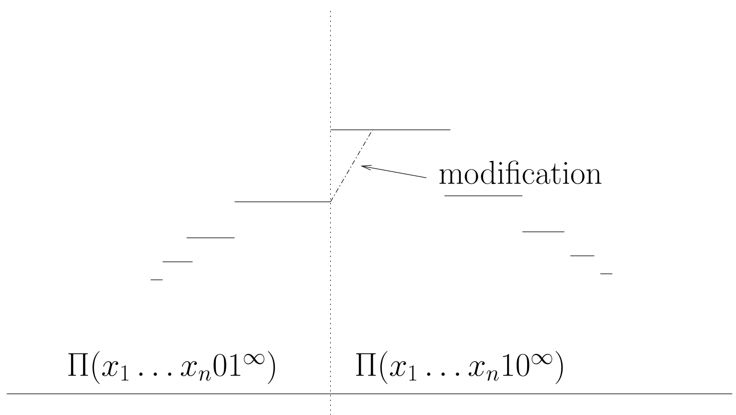


FIGURE 9. modification with two different suffixes

Positivity of W away from $\mathbb{K}$ follows from the positivity of V and the way of modifying it to get W. Clearly W vanishes on $\mathbb{K}$. □

The case $a < 1$. Continuing the proof of Theorem 5, the eigen-measure ν_a in Σ is a fixed point for the adjoint of the transfer operator for γ_1 (the pressure vanishes at γ_1) for the potential W. As the potential W is continuous and the shift is Markov, such a measure always exists. It is conformal in the sense that

$$\nu(\sigma(B)) = \int_B e^{\mathcal{P}(\gamma_1)+\gamma_1 W}\, d\nu_a = \int_B e^{\gamma_1 W}\, d\nu_a, \tag{20}$$

for any Borel set B on which σ is one-to-one. Since we have a phase transition at γ_1, $\mathcal{P}(\gamma_1) = 0$. Note also that W is positive everywhere except on $\mathbb{K}$ where it vanishes.

Now consider the measure $\Pi_*(\nu_a)$ and its distribution function

$$\theta_a(x) := \nu_a([0^\infty, \Pi(x))) = \nu_a([0^\infty, \Pi(x)]),$$

the last equality resulting from the fact that ν_a is non-atomic. We emphasize that Π maps the lexicographic order in Σ to the usual order on the unit interval $[0,1]$. This enables us to define *intervals* in Σ, for which we will use the same notation $[x,y]$.

Let us now compute the derivative of f_a define by

$$f_a := \theta_a \circ \Pi \circ \sigma \circ \Pi^{-1} \circ \theta_a^{-1}$$

at some point $x \in [0,1]$. For h very small we define y and y_h in $[0,1]$ such that $\Pi_*\nu_a([0,y]) = x$ and $\Pi_*\nu_a([0,y_h]) = x+h$. Also define $\underline{y}$ and $\underline{y_h}$ such that $\Pi(\underline{y}) = y$ and $\Pi(\underline{y_h}) = y_h$. Then we get

$$\begin{aligned}
\frac{f_a(x+h)-f_a(x)}{h} &= \frac{\nu_a([\sigma(\underline{y}),\sigma(\underline{y_h})])}{\nu_a([\underline{y},\underline{y_h}])} \\
&= \frac{\nu_a(\sigma([\underline{y},\underline{y_h}]))}{\nu_a([\underline{y},\underline{y_h}])} \\
&= \frac{1}{\nu_a([\underline{y},\underline{y_h}])}\int_{[\underline{y},\underline{y_h}]} e^{\gamma_1 W}\, d\nu_a \ \to_{h\to 0}\ e^{\gamma_1 W(\underline{y})}.
\end{aligned}$$

This computation is valid if $\Pi^{-1}(y)$ is uniquely determined (namely y is not dyadic). If y_h is dyadic for some h, then we choose for $\underline{y_h}$ the one closest to $\underline{y}$.

If y is dyadic, then the same can be done provided we change the preimage of y by Π depending on whether we compute left or right derivative. Nevertheless, the potential W is continuous at dyadic points, hence f_a has left and right derivative at every dyadic points and they are equal.

We finally get $f_a'(x) = e^{\gamma_1 W\circ\Pi^{-1}(x)}$ (this make sense also for dyadic points) and then $\log f_a'(x) = \gamma_1 W \circ \Pi^{-1}(x)$. Therefore f_a is $\mathcal{C}^1$ and as W is positive away from $\mathbb{K}$ and zero on $\mathbb{K}$, f_a is expanding away from $\widetilde{\mathbb{K}} := \theta_a \circ \Pi(\mathbb{K})$ and is indifferent on $\widetilde{\mathbb{K}}$. For $t \in [0,\infty)$, the lifted potential for $-t\log f_a'$ is $-t\gamma_1 W \circ \Pi^{-1}$, which has an ultimate phase transition for $t=1$ and $a \in (0,1)$. □

The case $a > 1$. Computations are similar to the case $a < 1$, except that we have to add the pressure for γ_1. The construction is the same, but the map f_a satisfies :

$$f_a'(x) = e^{\gamma_1 W\circ\Pi^{-1}(x)+\mathcal{P}(\gamma_1)}.$$

This extra term is just a constant and then, the thermodynamic formalism for $-t\log f_a'$ is the same that the one for $-t\gamma_1 W \circ \Pi^{-1}$.

3.5. **Unbounded potentials: Proof of Theorem 6.** We know from Subsection 2.5 that $\mathcal{R}$ fixes the potential V_u, defined in (9). In this section we set $g \equiv \alpha$, which gives $V_u = \alpha(k-1)$ on $(\sigma\circ H)^k(\Sigma)\setminus(\sigma\circ H)^{k+1}(\Sigma)$. For the thermodynamic properties of this potential, the interesting case is $\alpha < 0$ (see the Introduction before the statement of Theorem 6 and the Appendix).

Lemma 17. *Let $\alpha < 0$. Then $\int V_u d\mu \geqslant \int_\Sigma V_u d\mu_{\mathbb{K}} = 0$ for every shift-invariant measure probability μ.*

Proof. As in Lemma 6, the set $(\sigma\circ H)^k(\Sigma) = \sigma^{2^k-1}\circ H^k([00]\sqcup[10]\sqcup[01]\sqcup[11])$ consists of four 2^k+1-cylinders containing the points $1\rho_0$, $0\rho_0$, $1\rho_1$ and $1\rho_1$ respectively, and they are mapped into the two 2^k-cylinders containing ρ_0 and ρ_1. In other words, $(\sigma\circ H)^k(\Sigma) = \sigma^{-1}\circ H^k(\Sigma)$, and by Lemma 5, its next 2^k shifts are pairwise disjoint. Therefore $\mu_{\mathbb{K}}((\sigma\circ H)^k(\Sigma)) = 2^{-k}$ and $\mu_{\mathbb{K}}((\sigma\circ H)^k(\Sigma)\setminus(\sigma\circ H)^{k+1}(\Sigma)) = 2^{-(k+1)}$. Since $V_u = \alpha(k-1)$ on $(\sigma\circ H)^k(\Sigma)\setminus(\sigma\circ H)^{k+1}(\Sigma)$ this gives

$$\int V_u\, d\mu_{\mathbb{K}} = \alpha\sum_{k\geqslant 0}(k-1)2^{-(k+1)} = -\frac{\alpha}{2} + \alpha\sum_{k\geqslant 2} k2^{-(k+1)} = 0.$$

Again, since $\sigma^j((\sigma\circ H)^k(\Sigma)$ is disjoint from $((\sigma\circ H)^k(\Sigma))$ for $0<j<2^k$, its μ-mass is at most 2^{-k} for any shift-invariant probability measure μ. Since V_u is decreasing in k (for $\alpha<0$), we can minimize the integral $\int V_u\, d\mu$ by putting as much mass on $(\sigma\circ H)^k(\Sigma)$ as possible, for each k. But this means that the μ-mass of $(\sigma\circ H)^k(\Sigma)\setminus(\sigma\circ H)^{k+1}(\Sigma)$ becomes $2^{-(k+1)}$ for each k, and hence $\mu=\mu_{\mathbb{K}}$. □

Remark 6. *As a by-product of our proof, $\mu\left((\sigma\circ H)^k(\Sigma)\right)\leqslant 2^{-k}$ for any invariant probability μ and $k\geqslant 2$.* ■

For fixed $\alpha<0$, the integral $\int V_u\, d\mu$ is non-negative and we define for $\gamma\geqslant 0$

$$\mathcal{P}(\gamma) := \sup_{\mu\ \sigma-inv}\left\{h_\mu - \gamma\int V_u\, d\mu\right\}.$$

Proposition 18. *For any $\gamma\geqslant 0$ there exists an equilibrium state for $-\gamma V_u$.*

To prove this proposition, we need a result on the accumulation value $\liminf_{\varepsilon\to 0}\int V_u d\nu_\varepsilon$ if $\{\nu_\varepsilon\}_\varepsilon$ is a family of invariant probability measures.

Lemma 19. *Let ν_ε be a sequence of invariant probability measures converging to ν in the weak topology as $\varepsilon\to 0$. Let us set $\nu := (1-\beta)\mu+\beta\mu_{\mathbb{K}}$, where μ is an invariant probability measure satisfying $\mu(\mathbb{K})=0$ and $\beta\in[0,1]$. Then,*

$$\liminf_{\varepsilon\to 0}\int V_u\, d\nu_\varepsilon \geqslant (1-\beta)\int V_u\, d\mu.$$

Proof of Lemma 19. Let us consider an η-neighborhood O_η of $\mathbb{K}$ consisting of finite union of cylinders. Clearly $(\sigma\circ H)^j\subset O_\eta$ for $j=j(\eta)\geqslant 2$ sufficiently large (and $j(\eta)\to\infty$ as $\eta\to 0$).

Let ν_ε be an invariant probability measure. Following the same argument as in the proof of Lemma 17 and in particular Remark 6, we claim that

$$\int \mathbb{1}_{O_\eta} V_u\, d\nu_\varepsilon \geqslant -\frac{\alpha}{2}\nu_\varepsilon\left(O_\eta\setminus(\sigma\circ H)(\Sigma)\right) + \alpha\sum_{k\geqslant j} k2^{-(k+1)}$$

holds. Then we have

$$\begin{aligned}\int V_u\, d\nu_\varepsilon &\geqslant \int \mathbb{1}_{\Sigma\setminus O_\eta} V_u\, d\nu_\varepsilon - \frac{\alpha}{2}\nu_\varepsilon(O_\eta\setminus(\sigma\circ H)(\Sigma)) + \alpha\sum_{k\geqslant j} k2^{-(k+1)}\\ &\geqslant \int \mathbb{1}_{\Sigma\setminus O_\eta} V_u\, d\nu_\varepsilon + \alpha\sum_{k\geqslant j} k2^{-(k+1)}. \qquad (21)\end{aligned}$$

Note that $\mathbb{1}_{\Sigma\setminus O_\eta} V_u$ is a continuous function. Thus, $\lim_{\varepsilon\to 0}\int \mathbb{1}_{\Sigma\setminus O_\eta} V_u\, d\nu_\varepsilon$ exists and is equal to $\int \mathbb{1}_{\Sigma\setminus O_\eta} V_u\, d\nu = (1-\beta)\int \mathbb{1}_{\Sigma\setminus O_\eta} V_u\, d\mu$. As $\eta \to 0$, this quantity decreases and converges to $(1-\beta)\int V_u\, d\mu$ (here we use $\mu(\mathbb{K}) = 0$). Therefore, passing to the limit in (21) first in ε and then in η we get

$$\liminf_{\varepsilon\to 0}\int V_u\, d\nu_\varepsilon \geqslant (1-\beta)\int V_u\, d\mu.$$

□

Proof of Proposition 18. We repeat the argument given in the proof of Proposition 13 and adapt it as in [20]. Let ν_ε be a probability measure such that

$$h_{\nu_\varepsilon} - \gamma\int V_u\, d\nu_\varepsilon \geqslant \mathcal{P}(\gamma) - \varepsilon, \tag{22}$$

and let ν be any accumulation point of ν_ε. As V_u is discontinuous we cannot directly pass to the limit $\varepsilon \to 0$ and claim that the integral of the limit measure is the limit of the integrals. However, we claim that V_u is continuous everywhere but at the four points $0\rho_0$, $0\rho_1$, $1\rho_0$ and $1\rho_1$ (see and adapt the proof of Lemma 6). These points are in $\mathbb{K}$ and their orbits are dense in $\mathbb{K}$. We thus have to consider two cases.

- $\nu(\mathbb{K}) = 0$. Then a standard argument in measure theory says that we do not see the discontinuity, and passing to the limit as $\varepsilon \to 0$ in (22),
$$\mathcal{P}(\gamma) \geqslant h_\nu - \gamma\int V_u\, d\nu \geqslant \mathcal{P}(\gamma),$$
which means that ν is an equilibrium state.
- $\nu(\mathbb{K}) > 0$. In this case we can write $\nu = \beta\mu_{\mathbb{K}} + (1-\beta)\mu$, where μ is a σ-invariant probability satisfying $\mu(\mathbb{K}) = 0$ and β belongs to $(0, 1]$. Therefore
$$h_\nu = \beta h_{\mu_{\mathbb{K}}} + (1-\beta)h_\mu = (1-\beta)h_\mu.$$

Lemma 19 yields

$$\liminf_{\varepsilon\to 0}\int V_u\, d\nu_\varepsilon \geqslant (1-\beta)\int V_u\, d\mu. \tag{23}$$

Hence, passing to the limit in Inequality (22), Inequality (23) shows that

$$\mathcal{P}(\gamma) \leqslant (1-\beta)h_\mu - \gamma(1-\beta)\int V_u\, d\mu.$$

This last inequality is impossible if $\beta < 1$, by definition of the pressure. This yields that ν_ε converges to $\mu_{\mathbb{K}}$, and h_{ν_ε} converges to 0. Then (23) shows that $\mathcal{P}(\gamma) \leqslant 0$.

On the other hand $\mathcal{P}(\gamma) \geqslant 0$ because the pressure is larger than the free energy for $\mu_{\mathbb{K}}$, which is zero. Therefore $\mu_{\mathbb{K}}$ is an equilibrium state. □

In order to use Proposition 13 we need to check that V_u satisfies the hypotheses.

Lemma 20. *For every cylinder J which does not intersect $\mathbb{K}$, the potential V_u satisfies the local Bowen property* (12).

Proof. Actually, V_u satisfies a stronger property: if $x = x_0x_1\dots$ and $y = y_0y_1\dots$ are in J (a fixed cylinder with $J \cap \mathbb{K} = \emptyset$), if n is their first return time in J, and if $x_k = y_k$ for any $0 \leqslant k < n$, then $(S_nV_u)(x) = (S_nV_u)(y)$.

Assume that J is a k-cylinder, and assume without loss of generality that $n > k$. The coordinates x_j and y_j coincide for $0 \leqslant j < n$, but since J is a k-cylinder, we actually have

$$x_j = y_j \text{ for } 0 \leqslant j \leqslant n + k - 1.$$

We recall that V_u is constant on sets of the form $(\sigma \circ H)^m(\Sigma) \setminus (\sigma \circ H)^{m+1}(\mathbb{K})$. Therefore, to compute $V_u(z)$ for $z \in \Sigma$ we have to know which set $(\sigma \circ H)^m(\Sigma) \setminus (\sigma \circ H)^{m+1}(\mathbb{K})$ it belongs to. Lemma 6 shows that $z = z_0, z_1, \dots$ belongs to $(\sigma \circ H)^m(\Sigma) \setminus (\sigma \circ H)^{m+1}(\mathbb{K})$ if and only if $z_1 \dots z_{2^m}$ coincides with $[\rho_0]_{2^m}$ or $[\rho_1]_{2^m}$ and m is the largest integer with this property.

Let us now study $V_u(\sigma^j(x))$ (and $V_u(\sigma^j(y))$) for j between 0 and $n - 1$. We have to find the largest integer m such that $z_{j+1} \dots z_{j+1+2^m}$ coincides with $[\rho_0]_{2^m}$ or $[\rho_1]_{2^m}$. As J does not intersect $\mathbb{K}$, the word $x_n, \dots, x_{n+k-1}$ (which is also the word $y_n, \dots, y_{n+k-1}$) is not admissible for $\mathbb{K}$. Therefore, the largest m such that $z_{j+1} \dots z_{j+1+2^m}$ coincides with $[\rho_0]_{2^m}$ or $[\rho_1]_{2^m}$ satisfies

$$2^m \leqslant n - j + k - 1.$$

In other words, the integer m only depends on the digits where $\sigma^j(x)$ and $\sigma^j(y)$ coincide. Therefore $V_u(\sigma^j(x)) = V_u(\sigma^j(y))$. □

Remark 7. *An important consequence of Proposition 18 and Lemma 20 is that the conclusions of Proposition 13 hold. Although the potential V_u is not continuous (and in fact undefined at $\sigma^{-1}(\{\rho_0, \rho_1\})$), it satisfies the local Bowen condition, so that the discontinuity is "invisible" for the first return map to J. Proposition 18 then implies the existence of an equilibrium state. Furthermore, the critical $z_c(\gamma) \leqslant \mathcal{P}(\gamma)$. By a similar argument as used in* [20, Proposition 3.10] *it can be checked that the conclusion of Lemma 19 holds despite the discontinuity of V_u.* ■

Lemma 21. *Take $\alpha < 0$ and consider the potential V_u and some cylinder set J disjoint from $\mathbb{K}$. The critical parameter for the convergence of $(\mathcal{L}_{z,\gamma}\mathbb{I}_J)(x)$ satisfies $z_c(\gamma) \geqslant 2^{-e^{-\gamma\alpha+2}+1} > 0$ for every $\gamma \in \mathbb{R}$ and $x \in J$.*

Proof. We now explore the thermodynamic formalism of the unbounded fixed point V_u of $\mathcal{R}$ given by Equation 10. This potential is piecewise constant, and the value on cylinder sets intersecting $\mathbb{K}$ can be pictured schematically (with $\alpha = -1$) as follows:

ρ_0	1	0	0	1	0	1	1	0	0	1	1	0	1	0	0	1	$\cdots$
ρ_1	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	$\cdots$
V_u	1	0	1	-1	1	0	1	-2	1	0	1	-1	1	0	1	-3	$\cdots$

Here, the third line indicates the value of V_u at $\sigma^n(\rho_j)$ for $n = 0, 1, 2, 3, \dots$ and $j = 0, 1$. A single ergodic sum of length $b = 2^{k+1} - 2^{k-i}$(with $\alpha < 0$ arbitrary again) for points x in the same cylinder as ρ_0 or ρ_1 is

$$(S_bV_u)(x) = \sum_{j=0}^{2^{k+1}-2^{k-i}-1} V_u(\sigma^j(x)) = -\alpha(1+i).$$

Therefore, the contribution of a single excursionis

$$\Phi_{z,\gamma}(\xi) = \sum_{j=1}^{N} \sum_{k=0}^{b_j-1} \gamma\alpha(1-i_j) - b_j z$$

where $i = i_j$ is such that $b_j = 2^{k+1} - 2^{k-i}$. The contribution to $(\mathcal{L}_{z,\gamma}\mathbb{1}_J)(x)$ of one cluster of excursions then becomes $E_{z,\gamma}(\xi) \geqslant \sum_{N\geqslant 1} A^N$, where (assuming that $z \geqslant 0$)

$$\begin{aligned}
A = \sum_{\substack{\text{allowed} \\ b\geqslant 1}} e^{\gamma\alpha(1+i)-bz} &= \sum_{k\geqslant 1}\sum_{i=0}^{k-1} e^{\gamma\alpha(1+i)-(2^{k+1}-2^{k-i})z} \\
&\geqslant e^{\gamma\alpha} \sum_{k\geqslant 1}\sum_{i=0}^{k-1} e^{i\gamma\alpha - 2^{k+1}z} \\
&= e^{\gamma\alpha} \sum_{k\geqslant 1} \frac{1-e^{\gamma\alpha k}}{1-e^{\gamma\alpha}} e^{-2^{k+1}z} \geqslant e^{\gamma\alpha} \sum_{k\geqslant 1} e^{-2^{k+1}z}
\end{aligned}$$

Take an integer $M \geqslant e^{-\gamma\alpha+2}$ and $z = 2^{-(M+1)}$. Then taking only the M first terms of the above sum, we get the the entire sum is larger than

$$e^{\gamma\alpha} M e^{-2^{M+1}z} \geqslant e^{\gamma\alpha} e^{-\gamma\alpha+2} e^{-1} = e > 1.$$

Therefore, $A > 1$ and $\sum A^N$ diverges. Hence, the critical $z_c(\gamma) \geqslant 2^{-e^{-\gamma\alpha+2}+1} > 0$ for all $\gamma > 0$. □

Proof of Theorem 6. It is just a consequence of Proposition 13 that a phase transition can only occur at the zero pressure. This never happens, hence the pressure is analytic on $[0, \infty)$ and there is a unique equilibrium state for $-\gamma V_u$. □

Appendix: The Thue-Morse subshift and the Feigenbaum map

The logistic Feigenbaum map $f_{\text{q-feig}} : I \to I$ is conjugate to unimodal interval map f_{feig}, which solves a renormalization equation

$$f_{\text{feig}}^2 \circ \Psi(x) = \Psi \circ f_{\text{feig}}(x), \tag{24}$$

for all $x \in I$, where Ψ is an affine contraction depending on f_{feig}. Note that f_{feig} is not a quadratic map, but it has a quadratic critical point c. See [11] and [23, Chapter VI] for an extensive survey.

As a result of (24), f_{feig} is infinitely renormalizable of Feigenbaum type, *i.e.,* there is a nested sequence M_k of periodic cycles of 2^k-periodic intervals such that each component of M_k contains two components of M_{k+1}. The intersection $\mathcal{A} := \cap_{k\geqslant 0} M_k$ is a Cantor attractor on which f_{feig} acts as a dyadic adding machine. The renormalization scaling $\Psi : M_k \to M_{k+1}^{\text{crit}}$, where M_k^{crit} is the component of M_k containing the critical point, and on each M_k^{crit} we have $f_{\text{feig}}^{2^{k+1}} \circ \Psi = \Psi \circ f_{\text{feig}}^{2^k}$.

Furthermore, $\mathcal{A}$ coincides with the critical ω-limit set $\omega(c)$ and it attracts every point in I except for countably many (pre-)periodic points of (eventual) period 2^k for some $k \geqslant 0$. Hence $f_{\text{feig}} : I \to I$ has zero entropy, and the only probability measures it preserves are Dirac measures on periodic orbits and a unique measure on $\mathcal{A}$. This means that $f_{\mathrm{feig}} : I \to I$ is not very interesting from a thermodynamic point of view. However, we can extend f_{feig} to a quadratic-like map on the complex domain, with a chaotic Julia set $\mathcal{J}$ supporting topological entropy $\log 2$, and its dynamics is a finite-to-one quotient of the full two-shift (Σ, σ). Equation (24) still holds for the complexification $f_{\mathrm{feig}} : U_0 \to V_0$ (a quadratic-like map, to be precise), where Ψ is a linear holomorphic contraction, and $U_0 \Subset V_0$ are open domains in $\mathbb{C}$ such that U_0 contains the unit interval. Renormalization in the complex domain thus means that M_1^{crit} extends to a disks $U_1 \Subset V_1$ and $f_{\mathrm{feig}}^2 : U_1 \to V_1$ is a two-fold branched cover with branch-point c. The *little Julia set*

$$\mathcal{J}_1 = \{z \in U_1 : f_{\mathrm{feig}}^{2n}(z) \in U_1 \text{ for all } n \geqslant 0\}$$

is a homeomorphic copy under Ψ of the entire Julia set $\mathcal{J}$, but it should be noted that most points in U_1 eventually leave U_1 under iteration of f_{feig}^2: U_1 is not a periodic disk, only the real trace $M_1^{\mathrm{crit}} = U_1 \cap \mathbb{R}$ is 2-periodic. The same structure is found at all scales: $M_k^{\mathrm{crit}} = U_k \cap \mathbb{R}$, $U_k \Subset V_k$ and $f_{\mathrm{feig}}^{2^k} : U_k \to V_k$ is a two-fold covering map with little Julia set

$$\mathcal{J}_k := \{z \in U_1 : f_{\mathrm{feig}}^{2^k n}(z) \in U_1 \text{ for all } n \geqslant 0\} = \Psi(\mathcal{J}_{k-1}).$$

To explain the connection between $f_{\mathrm{feig}} : \mathcal{J} \to \mathcal{J}$ and symbolic dynamics, we first observe that the *kneading sequence* ρ (*i.e.,* the itinerary of the critical value $f_{\mathrm{feig}}(c)$) is the fixed point of a substitution

$$H_{\mathrm{feig}} : \begin{cases} 0 \to 11, \\ 1 \to 10. \end{cases}$$

Let $\Sigma_{\mathrm{feig}} = \overline{\mathrm{orb}_\sigma(\rho)}$ be the corresponding shift space. If we quotient over the equivalence relation $x \sim y$ if $x = y$ or $x = w0\rho$ and $y = w1\rho$ (or vice versa) for any finite and possibly empty word w, then $\Sigma_{\mathrm{feig}} \sim$ is homeomorphic to $\mathcal{A}$, and the itinerary map $i : \mathcal{A} \to \Sigma_{\mathrm{feig}}/\sim$ conjugates f_{feig} to the shift σ.

To make the connection with the Thue-Morse shift, observe that the sliding block code $\pi : \Sigma \to \Sigma$ defined by

$$\pi(x)_k = \begin{cases} 1 & \text{if } x_k \neq x_{k+1}, \\ 0 & \text{if } x_k = x_{k+1}, \end{cases}$$

is a continuous shift-commuting two-to-one covering map. The fact that it is two-to-one is easily seen because if $x_k = 1 - y_k$ for all k, then $\pi(x) = \pi(y)$. Surjectivity can also easily be proved; once the first digit of $\pi^{-1}(z)$ is chosen, the following digits are all uniquely determined. It also transforms the Thue-Morse substitution H into H_{feig} in the sense that $H_{\mathrm{feig}} \circ \pi = \pi \circ H$. For the two Thue-Morse fixed points of H we obtain

$$\pi(\rho_0) = \pi(\rho_0) = \rho = 1011101010111011101110101011101 0\dots$$

Figure 10 summarizes all this in a single commutative diagram.

The Cantor set $\mathbb{K}$ factorizes over Σ_{feig} and hence over the Cantor attractor $\mathcal{A}$. The intermediate space $\mathbb{L}$ factorizes over the real *core* $[c_2, c_1]$ in the Julia set $\mathcal{J}$ and we can characterize its symbolic dynamics by means of a particular order relation. Namely, itineraries $i(z)$ of $z \in [c_2, c_1]$ are exactly those sequences that satisfy

$$\sigma(\rho) \leqslant_{pl} \sigma^n \circ i(z) \leqslant_{pl} \rho \quad \text{ for all } n \geqslant 0.$$

$$\begin{tikzcd}
\rho_0, \rho_1 \in \mathbb{K} \subset \mathbb{L} \subset \Sigma \arrow[r, "H"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi"] \\
\rho \in \Sigma_{\text{feig}} \subset \pi(\mathbb{L}) \subset \Sigma \arrow[r, "H_{\text{feig}}"] \arrow[d, "\sim"'] & \Sigma \arrow[d, "\sim"] \\
\Sigma/\sim \arrow[r, "H_{\text{feig}}"] & \Sigma/\sim \\
f_{\text{feig}}(0) \in \mathcal{A} \subset [c_2, c_1] \subset \mathcal{J} \arrow[r, "\Psi"] \arrow[u, "i"] & \mathcal{J}_1 \arrow[u, "i"']
\end{tikzcd}$$

Thue-Morse subst. $H : \begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases}$

$\pi(x)_k = \begin{cases} 1 & \text{if } x_k \neq x_{k+1} \\ 0 & \text{if } x_k = x_{k+1} \end{cases}$

Feigenbaum subst. $H_{\text{feig}} : \begin{cases} 0 \to 11 \\ 1 \to 10 \end{cases}$

Equivalence relation $w1\rho \sim w0\rho$

i is the itinerary map

$\Psi : \mathcal{J} \to \mathcal{J}_1$ is renormalization scaling
$\mathcal{J}_1$is the first little Julia set
Feigenbaum map f_{feig} on Julia set $\mathcal{J}$

FIGURE 10. Commutative diagram linking the Thue-Morse substitution shift to the Feigenbaum map. Further commutative relations:
π is continuous, two-to-one and $\sigma \circ \pi = \pi \circ \sigma$.
$i : [c_2, c_1] \to \pi(\mathbb{L})/\sim$ is a homeomorphism and $\sigma \circ i = i \circ f_{\text{feig}}$.
$\sigma^2 \circ H = H \circ \sigma$, $\sigma^2 \circ H_{\text{feig}} = H_{\text{feig}} \circ \sigma$ and $f_{\text{feig}}^2 \circ \psi = \psi \circ f_{\text{feig}}$.

Here $\leqslant_{pl}$ is the *parity-lexicographical order* by which $z <_{pl} z'$ if and only if there is a prefix w such that

$$\begin{cases} z = w0\dots, \quad z' = w1\dots \text{ and } w \text{ contains an even number of 1s,} \\ z = w1\dots, \quad z' = w0\dots \text{ and } w \text{ contains an odd number of 1s.} \end{cases}$$

On the level of itineraries, the substitution H_{feig} plays the role of the conjugacy Ψ:

$$i \circ \Psi(x) = H_{\text{feig}} \circ i(x) \qquad \text{for all } x \in [c_2, c_1].$$

Also let $\leqslant_l$ denote the usual lexicographical order.

Lemma 22. *Let* $[0]$ *and* $[1]$ *denote the one-cylinders of* Σ. *The map* $\pi : ([0], \leqslant_l) \to (\Sigma, \leqslant_{pl})$ *is order preserving and* $\pi : ([1], \leqslant_l) \to (\Sigma, \leqslant_{pl})$ *is order reversing.*

Proof. First we consider $[0]$ and let $w = 0^n$, then $w0 \dots <_l w1 \dots$ and

$$\pi(w0\dots) = 0^n \dots \leqslant_{pl} 0^{n-1}1 \dots = \pi(w1\dots). \tag{25}$$

Now if we change the k-th digit in w (for $k \geqslant 2$), then still $w0 <_l w1$ and both the k-th and $k-1$-st digit of $\pi(w\dots)$ change. This does not affect the parity of 1s in $\pi(w)$ and so (25) remains valid. Repeating this argument, we obtain that π is order-preserving for all words w starting with 0.

Now for the cylinder $[1]$ and $w = 10^{n-1}$, we find $w0 \dots <_l w1 \dots$ and

$$\pi(w0\dots) = 10^{n-1} \dots \geqslant_{pl} 10^{n-2}1 \dots = \pi(w1\dots).$$

The same argument shows that π reverses order for all words w starting with 1. $\square$

This lemma shows that $\pi^{-1} \circ i([c_2, c_1])$ consists of the sequence s such that for all n,

$$\begin{cases} \sigma(\rho_1) \leqslant_l \sigma^n(s) \leqslant_l \rho_1 & \text{if } \sigma^n(s) \text{ starts with } 1, \\ \rho_0 \leqslant_l \sigma^n(s) \leqslant_l \sigma(\rho_0) & \text{if } \sigma^n(s) \text{ starts with } 0. \end{cases}$$

However, the class of sequence carries no shift-invariant measures of positive entropy, and the thermodynamic formalism reduces to finding measures that maximize the potential. The measure supported furthest away from $\mathbb{K}$ is the Dirac measure on $\overline{01}$ (with $\pi(\overline{01}) = \overline{1}$).

Instead, if we look at the entire Julia set $\mathcal{J}$, the combination of π and the quotient map do not decrease entropy, and the potential $-\log|f'_{\text{feig}}|$ has thermodynamic interest for the complexified Feigenbaum map $f_{\text{feig}} : \mathcal{J} \to \mathcal{J}$. Since Ψ is affine, differentiating (24) and taking logarithms, we find that

$$\mathcal{R}_{\text{feig}}(\log|f'_{\text{feig}}|) := \log|f'_{\text{feig}}| \circ f_{\text{feig}} \circ \psi + \log|f'_{\text{feig}}| \circ \Psi = \log|f'_{\text{feig}}|,$$

so $V_{\text{feig}} := \log|f'_{\text{feig}}|$ is a fixed point of the renormalization operator $\mathcal{R}_{\text{feig}}$ mimicking $\mathcal{R}$. Furthermore, since $U_k = \Psi^{k-1}(U_1)$, its size is exponentially small in k and hence there is some fixed $\alpha < 0$ such that $V_{\text{feig}} \approx \alpha(k-1)$ on $U_k \setminus U_{k+1}$. Since $U_k \setminus U_{k+1}$ corresponds to the cylinder $(\sigma \circ H)^{k-1} \setminus (\sigma \circ H)^k$, the potential V_u from Section 2.5 is comparable to V_{feig}. As shown in Section 3.5, V_u exhibits no phase transition.

The following proposition for complex analytic maps is stated in general terms, but proves the phase transition of Feigenbaum maps in particular.

Proposition 23. *Let $f : \mathbb{C} \to \mathbb{C}$ be an n-covering map without parabolic periodic points such that the omega-limit set $\omega(\text{Crit})$ of the critical set is nowhere dense in its Julia set $\mathcal{J}$, and such that there is some $c \in \text{Crit}$ such that $f : \omega(c) \to \omega(c)$ has zero entropy and Lyapunov exponent. Then for $\phi = \log|f'|$ and every $\gamma > 2$, $\mathcal{P}(-\gamma\phi) = 0$.*

Proof. As f has no parabolic points, $\lambda_0 := \inf\{|(f^n)'(p)| \,:\, p \in \mathcal{J} \text{ is an } n\text{-periodic point}\} > 1$. Obviously, all the invariant measures μ supported on $\omega(c)$ have $h_\mu - \gamma \int \log|f'| d\mu = 0$, so $\mathcal{P}(-\gamma\phi) \geqslant 0$.

To prove the other inequality, we fix $\gamma > 2$ and for some f-invariant measure μ, we choose a neighborhood U intersecting $\mathcal{J}$ but bounded away from $\text{orb}(\text{Crit})$ such that $\mu(U) > 0$. We can choose $\text{diam}(U)$ so small compared to the distance $d(\text{orb}(\text{Crit}), U)$ that $K^{\gamma-1} < \lambda_0^{\gamma-2}$ where K is the distortion constant in the Koebe Lemma, see [24, Theorem 1.3]. Since $K \to 1$ as $\kappa := \text{diam}(U)/d(\text{orb}(\text{Crit}), U) \to 0$, we can satisfy the condition on K by choosing U small enough.

Let $F : \cup_i U_i \to U$ be the first return map to U. Each branch $F|_{U_i} = f^{\tau_i}|_{U_i}$, with first return time $\tau_i > 0$ can be extended holomorphically to $f^{\tau_i} : V_i \to f^{\tau_i}(U_i)$ where $f^{\tau_i}(V_i)$ contains a disc around $f^{\tau_i}(U_i)$ of diameter $\geqslant d(\text{orb}(\text{Crit}), U) \geqslant \text{diam}(f^{\tau_i}(U_i))/\kappa$. Hence the Koebe Lemma implies that the distortion of $f^{\tau_i}|_{U_i}$ is bounded by $K = K(\kappa)$. Furthermore, since each U_i contains a τ_i-periodic point of multiplier $\geqslant \lambda_0$, we have $\text{diam}(U_i)/\text{diam}(U) \leqslant$

K/λ_0. Therefore, for any $x \in U$,

$$\begin{aligned}
\mathcal{L}_{0,\gamma}(\mathbb{1}_J)(x) &= \sum_{i, \exists x' \in U_i \ F(x')=x} |F'(x')|^{-\gamma} \\
&\leqslant \sum_i K \left(\frac{\mathrm{diam}(U_i)}{\mathrm{diam}(U)} \right)^{\gamma} \\
&\leqslant \sum_i K \, \frac{\mathrm{area}(U_i)}{\mathrm{area}(U)} \left(\frac{K}{\lambda_0} \right)^{\gamma-2} \\
&\leqslant K^{\gamma-1} \lambda_0^{-(\gamma-2)} \sum_i \frac{\mathrm{area}(U_i)}{\mathrm{area}(U)}.
\end{aligned}$$

Since the regions U_i are pairwise disjoint, the sum in the final line $\leqslant 1$, so our choice of K gives that the above quantity is bounded by 1. Therefore the radius of convergence $\lambda_{0,\gamma} \leqslant 1$. Taking the logarithm and using Abramov's formula, we find that the pressure $\mathcal{P}(-\gamma\phi) \leqslant 0$. □

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